Geometry :: 6-9

MONTESSORI TEACHERS COLLECTIVE (MOTEACO.ORG)

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Introduction to Geometry

Maria Montessori described it as "psycho-geometry" and defined it so the "measurement of the earth together with the consciousness of the reciprocal relationship between Man and the objects of the environment, and between the objects themselves."

Maria Montessori extends this definition beyond the etymology (geometry: Greek *ge* the earth, Mother Earth from mythology, and *metron* measure, measurement of a dimension that is so vital to our lives.) Her definition especially this consciousness of geometry is so practical and so attached to reality for we are all a part of this.

Many refer to geometry as abstract, thus giving it to the child at a much later age and with the attitude that it is far removed from us and from reality.

Geometry, as it will be dealt with here, are the fundamental concepts of Euclidean geometry.

Chapter One – Sensorial Exploration of Shapes

The Geometry Cabinet

INTRODUCTION

With relation to the senses, Maria Montessori has extended the number of senses from five to seven. To the senses of smell, taste, sight, hearing and touch, she added the stereognostic sense, (the knowledge of 3-dimensionality) and the basic sense (the sense of mass, that is, of heaviness or lightness). The visual and stereognostic senses are directly related to the following work in geometry. Maria Montessori has also identified three different aspects of education of the visual sense: according to size, form and color. In geometry we will deal with visual education according to size and form, thus eliminating color. If the child did not have previous Children's House training, this visual education must be offered differently, because it is really only pertinent to a younger age.

PLANE INSETS

Materials:
...Geometry cabinet
...Additional insets, including pictures of the figures
...2 of the 3 boxes of pictures of the figures:
...entire figure "surface" shaded; the fine "contour" margin of the figure
...Reading labels
...box of command cards

Description of Materials:
geometry cabinet- The presentation of this material follows the order in which the drawers are arranged. Since the presentations differ from Children's House to the elementary school, so the order

of the drawers and the arrangement of the contents of each drawer differs from Children's House to the Elementary school.

Order:
Presentation tray - 0 comes first at both levels.

The names of the drawers in the Children's House and their order is:

1- circles; 2 - rectangles; 3 - triangles; 4 - polygons; and 5 - different figures.

At the elementary level the names and order are:

1 - triangles; 2 - rectangles; 3 - regular polygons; 4 - circles; and 5 - other figures.

At the Children's House level, the children worked directly for the education of the visual sense, and only indirectly to learn the geometric figures. In the elementary school what was a sensorial exploration becomes a linguistic exploration via etymology. What was an indirect approach to geometry becomes an actual study of geometry.

Therefore in elementary the drawer of triangles comes first because the triangle is the first polygon we can construct in reality, having the least number of sides. The second drawer logically follows as the quadrilaterals, specifically rectangles. Regular polygons follow beginning with the five-sided figure progressing to ten sides. circles follow, because a circle is the limit of a regular polygon having an infinite number of sides.

From Children's House to elementary the order has changed: from easiest to most difficult, to: from threes sides to an infinite number of sides. This correlates with the change from seeing, touching, and naming to a focus on etymology and reasoning.

Arrangement:
The presentation tray contains the three fundamental figures of geometry, that is the only regular figures. The equilateral triangle is the only regular triangle. The square is the only regular quadrilateral. The circle is the limit of all regular polygons having an infinite number of sides. the triangle is "the constructor of reality". For every plane figure can be decomposed into triangles, just as all solids can be decomposed into tetrahedrons. The square is the "measurer of surfaces" just as the cube is the measurer of solids. The circle is the measurer of angles. In Children's House. In the Children's House the arrangement is square (left), circle (top), triangle (right). In elementary the arrangement is triangle (left), square (top), circle (right).

The triangle tray examines triangles according to their sides on top; the bottom three examine triangles according to their angles, at both levels. In the Children's House the order is (top - from left to right): equilateral, isosceles, scalene (bottom - from left to right), acute-angled, right-angled, obtuse-angled. In elementary (top - from left to right): scalene, isosceles, equilateral (bottom - from left to right), right-angled, obtuse-angled, acute- angled.

In the rectangle tray, the base of the smallest figure is 5 cm. which is 1/2 the base of the largest which is a square. In Children's House the order is largest to smallest, elementary the reverse.

The regular polygon tray is ordered identically at both levels, progressing from five to ten sides. It is understood that these are the regular polygons having more than four sides, since the equilateral triangle and the square (first tray) are also regular polygons.
In the circle tray, the diameter of the smallest is 5 cm.; the diameter of the largest is 10 cm. It is ordered from largest to smallest in the Children's House and the reverse in elementary.

The arrangement in the other figure drawer is the same for both levels: trapezoid, rhombus, quatrefoil, oval, ellipse, and curvilinear triangle (Reuleaux triangle).

Additional insets for the geometry cabinet:

Two triangles: acute-angled scalene triangle, obtuse-angled scalene triangle

Eight quadrilaterals:
common quadrilateral (four different sides and four different angles)
common parallelogram (opposite sides are parallel and equal)
four trapezoids
equilateral trapezoid
(constructed from three equilateral triangles)
scalene trapezoid
right-angled trapezoid
obtuse-angle trapezoid (two obtuse angles opposite)
Two deltoids or kites: one with unequal diagonals
one with equal diagonals
Two quatrefoils: quadrilobed
epi-cycloid

Including surface cards for each.

Note: Ten dominates all of the plane insets:

Presentation tray: triangle sides - 10cm.; square sides - 10 cm.; circle diameter - 10 cm.

Triangles: Hypotenuse of the obtuse-angled triangle - 10 cm.

Rectangles: Height of each - 10 cm.

Regular polygons: All can be inscribed in a 10 cm. diameter circle

Circles: Diameter of largest - 10 cm.

Other figures: Trapezoid base, short diagonal in rhombus, distance between opposite lobes in quatrefoil, distance between two opposite cusps in oval and ellipse, base of triangle used to construct curvilinear triangle, all - 10 cm.

Extra figures: Triangles, diagonal of parallelogram, equilateral trapezoid base all 10 cm.
Distance between points on adjacent lobes of quadrilobed quatrefoil, and between opposite lobes of epi-cycloid - 10 cm.
No 10 cm. exists in the common quadrilateral, deltoids and the last three trapezoids.

THE GEOMETRY CABINET

Introduction:
In this second presentation of the geometry cabinet (first being in CH) the visual memory is aided by etymology, and no longer by the tactile sense. therefore the emphasis on that element is eliminated. Instead the emphasis is placed on etymology - the heart of our language.

Presentation tray

Materials:
...Appropriate drawer
...Three reading labels - "triangle", "square/quadrangle", and "circle"

Presentation: With only the tray on the table, the teacher takes out the triangle and identifies it.. this is a triangle. The child is asked to identify the angles and count them (triangle: Latin *tres*, *tria* - three and *angulus* - an angle; thus *triangulum* - triangle). Triangle means three angles. Place the inset in its frame in the drawer.

The teacher isolates the square and identifies it (square: Old French *esquarre, esquerre* <Latin *ex* - out, and *squadra* - square; thus to make square>). It is such an old word that the etymology doesn't help us as much. Put the square back. Isolate the circle and identify it (circle: Latin *circulus* - a diminutive of *circus* - a circle). Again the etymology doesn't help us because this shape has been called a circle as far back in time as we know.

As all three inset are placed on the table, review the first period. Rearrange the order and continue with the second and third periods. Invite the child to place the insets in their frames.

Exercise: Give the child the reading labels to place on the insets in their frames: triangle, circle, square/quadrangle. Note: The word quadrangle is not used at this point.

TRIANGLES

Materials: Reading labels - "scalene triangle", "isosceles triangle", "equilateral triangle", "right-angled triangle", "obtuse- angled triangle", "acute-angled triangle"

Presentation: Take out the first triangle in the first row. Invite the child to identify the three sides and observe whether the sides are alike or different. all three sides are different, this is a scalene triangle. Relate the story of the farmer and the ladder he used to pick fruit from his trees. Unlike the ladders we use today, the rungs of this ladder were all different lengths. These ladders are still used today in lesser developed countries. Just as all the rungs are different lengths, the sides of this triangle are all different lengths (scalene: Latin *scala*, usually plural *scalae* - ladder, flight of steps or Greek: *skalenas* - limping, uneven).

Isolate the second triangle in the first row. Invite the child to carefully observe its sides - two are alike. This is an isosceles triangle (isosceles: Greek *isos* - equal, and *sceles* - legs; thus having equal legs). Here it means two equal legs, or sides.

Isolate the third triangle. By observing and turning the inset in its frame, the child sees that all of the sides are the same. This is an equilateral triangle (equilateral: Latin *aequus* - equal, and *latus*, *lateris* - a side; thus having equal sides). Place the three insets on the table and do a three period lesson.

Isolate the first triangle in the second row. Identify the right angle. This is a right angle, it is erect. This is a right-angled triangle. How many right angles does it have? Only one.

Isolate the second triangle. Identify the obtuse angle. Obtuse means dull. This is an obtuse-angled triangle. Count the obtuse angles... only one.

Isolate the third triangle. All of these angles are smaller than the right angle. They are acute angles. Acute means sharp, pointed. (feel how it is sharper than the right or obtuse angles). This is an acute-angled triangle. How many acute angles does it have? Three.
Bring out the three triangles and review the first period. The triangle must have one right angle to be a right-angled triangle... and so on. Second and third periods follow. Give the child the reading labels.

RECTANGLES

Materials: Reading labels: five "rectangle" and one "rectangle/square"

Presentation: Isolate the first inset. Identify it and give etymology (rectangle: Latin *rectus*- right, and *angulus* - an angle; thus having all right angles. Invite the child to identify the other rectangles as they are isolated.

Isolate the last inset. This is also a rectangles because it has all right angles, but it is also a square. Do a three period lesson and give the child the reading labels.

REGULAR POLYGONS

Materials: Reading labels: "pentagon", "hexagon", "octagon", "nonagon", "decagon", and a series of ten cards: "</angulus", "3/tria-", "4/quatuor-", "5/pente", "6/hex-", "7/hepta-", "8/okto-", "9/nonus or ennea", "10/deca-", "n/polys-"
Drawer 3 and the frame and inset of triangle and square from the presentation tray.

Presentation: Position the two extra insets to the left of the drawer in line with the top row. Isolate the triangle. Invite the child to identify an angle. Identify one on the square also. Isolate the decagon and invite the child to identify an angle. Feel it and compare it to the triangle and square. This angle is less sharp than the angles f the triangle.

Present the symbol card which represents angle (<). Identify the angles on the triangle and count them. Place the 3 card and the angle card side by side over the inset frame. Continue with each of the other figures, counting the angles, and placing the corresponding numeral card with the angle card. Since there is only one angle card, it floats from one inset to the next as needed.

Isolate the triangle inset and the two cards 3 <. The child identifies the figure and gives the meaning of its name. Then turn over the cards reading the Latin words which were made into a compound word to get triangle. Return the inset to its frame with its number card.
Isolate the square inset and cards: 4 <. Turn over the cards to find that 4 angles was quatuor angulus, from which our word quadrangle was derived.

Go on naming the other figures in this way using the Greek word for angle - *gonia*. Note: *nonus* - ninth, and *ennea* - nine.

After ten we have no more figures in our materials. Imagine a figure with any number of sides... 15, 20, 100, any figure with more than three sides. We can indicate this number by n. Bring out the card and place next to it the angle sign. turn over the cards: polys - many, and gonia - angle. Any figure that has more than three sides is a polygon. All of these figures we've examined up to now are polygons.
Beginning with the triangle turn all of the figures in their frames to show that the sides and angles are equal. All of these are "regular polygons". Name each figure: regular triangle is an equilateral triangle; a regular quadrangle is a square; a regular pentagon; a regular hexagon... and so on. Do a three-period lesson and give the reading labels.

CIRCLES

Materials: Reading labels: 4 "circle", 1 "circle (smallest)", 1 "circle (largest)"

Presentation: The child identifies all as circles and puts out the reading labels.

OTHER FIGURES

Materials: Reading labels: "trapezoid", "rhombus", "quatrefoil", "oval", "ellipse", "curvilinear triangle" or "Reuleaux triangle"
Frame of the circle inset (for presentation of ellipse)

Presentation: Isolate the trapezoid and identify it (trapezoid: Greek: *trapezion* - a little table). In order to understand why this figure has its name we must go back in time to see what a table of the Greeks looked like. Nowadays our tables don't look trapezoidal. Some Spanish tables have two legs but still not trapezoidal. The Greek table was like a Spanish table because it had two legs, yet it was more stable because the legs were inclined.

Isolate the rhombus and identify it. This is a rhombus (rhombus: Greek: *rhombos* - magic wheel, top) In ancient Greece, in the city of Athens, during a religious procession through the streets, a priest walked along with a cane (rod) raised over his head. At the end of the cane there was a cord attached, and at the end of the cord there was a rhombus-shaped figure attached. He rotated the cane in the air as he walked causing this figure to spin around like a top, making a characteristic sound. This was part of a religious ritual.
Isolate the quatrefoil and identify it (quatrefoil: Old French *quatre* - four, and *foil* - leaf). This figure has the shape of a four-leaf clover, considered a sign of good luck.

Isolate the oval and identify it (oval: French *ovale* <Latin *ovum*> - egg). This figure has the shape of an egg.

Isolate the ellipse and identify it (ellipse: Greek *elleipsis* - an omission or defect <*elleipo* - to leave out>). What has been left out? Think of the ideal figure, the circle. Place the inset of the ellipse in the circle frame and it is easy to what is missing. This is also the shape of the path that the earth follows around the sun.

Isolate the Reuleaux triangle and identify it (curvilinear: Latin *curvus* - curved, and *linear* - a line). This triangle has three sides which are curved lines. It is named after a man name Reuleaux who discovered the properties of this shape. He found that a drill bit made in this shape will make square holes.

Give three-period lesson and give the reading labels.

Age: 6 years and on

Aim: Knowledge of the geometric figures and their relative exact nomenclature.

Constructive Triangles - First Series

Note: The triangle, the smallest figure in reality is the constructor of all other figures in reality. The tetrahedron, the smallest solid in reality constructs all of the other solids in space.

Materials:
Box 1-
Two yellow equilateral triangles
Two yellow, two green right-angled isosceles triangles
Two yellow, two green, two gray right-angled scalene triangles
One red smaller right-angled scalene triangle

On red obtuse-angled scalene triangle
(Each triangle has a black line along one side)

Box 2 -
Two blue equilateral triangles
Two blue right-angled isosceles triangles
Two blue right-angled scalene triangles
One blue obtuse-angled scalene triangle
One blue right-angled scalene triangle (corresponds to the red triangle from Box 1)

Box 3 -
Twelve blue right-angled scalene triangles with no black lines. The angles measure 30, 60, and 90 degrees.

FIRST BOX

Presentation: Invite the child to remove the triangles from the box, and then sort the triangles according to shape. Having done this, ask the child to separate each pile according to color, resulting in various piles of triangles having both shape and color in common. Isolate the two red ones to be used later.

The teacher takes the pile of two equilaterals and separates them in such a way that the two black lines are facing each other. Watch, these black lines are like a magnet. Slide the two triangles together so that the black lines meet. Invite the child to do the same, leaving the joined triangles in place.

Identify the figures that have been constructed: a yellow rhombus, a green square, and a gray rectangle. The teacher identifies the other three figures as common parallelograms (parallelogram: Greek *parallelogramium* <*parallelos*, parallel, and *grame*, figure>). therefore a parallelogram is a plane figure having parallel sides. By simultaneously running two fingers along two parallel sides, the teacher gives a sensorial impression of parallel. We also call them common parallelograms to differentiate them from the square, rectangle, and rhombus which could also be considered parallelograms. The child names each figure as they are indicated by the teacher.

SECOND BOX - CONSTRUCTION OF QUADRILATERALS

Presentation: Invite the child to sort the triangles by shape. As before, set aside the two small triangles which correspond to the red ones of the first box.

Isolate the two equilateral triangles and invite the child to form all of the possible quadrilaterals. try as he might, he can only form one. The child identifies it as the rhombus.

Leaving the rhombus intact, the teacher takes the two right-angled isosceles triangles and forms the possible figures. The child identifies the figures as they are made. there are two: the square and the common parallelogram.

The child may see two different parallelograms. Trace one on a sheet of paper. Form the other and superimpose it. The second parallelogram doesn't fit inside the contours of the first. Trace the second parallelogram and cut out the two figures. By placing the cut-outs back to back, we can see that one is the mirror image of he other, therefore they are the same parallelogram.

The child is invited to form the possible quadrilaterals with the two right-angled scalene triangles. The child identifies the three:

rectangle, common parallelogram, a different parallelogram.

One by one, isolate each type of triangle, ask the child to classify the triangle according to its sides, and ask, "Of how many different lengths are the sides of this triangle"? Conclude that with a triangle whose sides have all one measure, we can form only one figure - the rhombus. With a triangle whose sides have two different measures, we can form to figures - square and common parallelogram. With a triangle whose sides have three different measures, we can form three figures - rectangle and two parallelograms.

Direct Aim: To give the relationship between the number of different lengths of the sides and the number of figures which can be possibly constructed.

SECOND BOX - HEADS & TAILS

Presentation: Every plane figure, like a coin, has two sides. One side is called obverse or heads; it is the side which has a face or the principle design. When you turn it over, you have the reverse side or tails. All of these figures have an obverse (blue) and a reverse (natural wood or white) side.

Isolate the two equilateral triangles. Invite the child to form as many figures as possible. As before, he can make only one. Suggest that he tries with one obverse and one reverse side. It won't help. There is only one figure he can make.

With the two isosceles triangles he makes the two possible quadrilaterals with the obverse sides. Invite the child to make a triangle. The child classifies the triangle: isosceles. By turning one triangle to its reverse side, the child can make no new figures.

With the two scalene triangles, the child tries to form all possible quadrilaterals first with the two obverse sides (yielding the same three figures as before) and then with one reverse side. The child is able to form a new figure: a kite (or a deltoid, having the form of the Greek capital letter Delta).

Invite the child to make triangles, first with obverse sides (yielding none) and then with one reverse side. The child classifies the triangles he makes: acute-angled isosceles and obtuse-angled isosceles triangles.

TRAPEZOID

Presentation: With the two small red triangles, invite the child to unite the triangles along the black lines and identify the figure obtained - a trapezoid.

Using the two corresponding blue triangles, invite the child to form the figure which he already knows and to identify it - a trapezoid. Continue making other quadrilaterals using only the obverse sides. The teacher identifies the figure obtained. since it has four sides we can call it a quadrilateral. It is a concave quadrilateral - a boomerang (it may also be called a re-entrant).

Invite the child to turn over one triangle to form any other figures, quadrilaterals or triangles. The quadrilateral is called a common quadrilateral. The triangle is an obtuse-angled isosceles triangle. (Note: This triangle has great importance in the later study of the area of a trapezoid.) Recall the figures formed by these triangles; there are four.

Age: After 6 years

Aim: Exploration of the triangle as the constructor of triangles and quadrilaterals.

THIRD BOX

Presentation: Isolate one triangle. Ask the child to identify each of the angles, and the biggest and smallest angles. This angle which is neither smallest nor biggest we can call the medium angle. The child names each angle: smallest angle, medium angle, biggest angle.

First star: Let's unite all the triangles by their smallest angle. The teacher positions a few and allows the child to continue. How many points does this star have? Twelve. with all of the triangles at our disposal, we can make only one star with twelve points.

Second star: Let's unite all the triangles by the medium angles. How many points does this star have? Six. Try to make another star with the triangles that are left. With all the triangles at our disposal, we can make two stars with six points.

Third star: Let's unite all of the triangles by the largest angle. How many points does this star have? Four. This symbol is very famous; it is the star of Saint Brigid, the patron saint of Ireland. Try to make another star like this. With all the triangles at our disposal, we can make three stars with four points.

Aim: Use of the triangle as a constructor to indirectly demonstrate the following:
30o x 12 triangles = 360o 60o x 6 triangles = 360o 90o x 4 = 360o
360o / 30o = 12 tris. 360o / 60o = 6 tris. 360o / 90o = 4 tris.
360o / 12 triangles 360o / 6 = 60o 360o / 4 = 90o

Related activities:

1. Construct the first star. Notice that the triangles meet at a point in the center. We must divide this star into two equal parts, leaving six triangles on one side and six on the other. Many possibilities exist; simply choose on and slide the triangles away to leave a gap.
We want to make the point at the top of one side meet the point on the top of the other. Slide one half along and then towards the other to make the two points meet at the top. We see that they have met at the bottom also, and where there was a point in the center there is now a line segment.

Again divide the figure in half, this time along the other side of the triangle which was displaced before. Separate the two halves to leave a gap. Identify the two points at the top and bottom which should meet. Slide one half into position. We see that a quadrilateral (a rhombus) has been created at the center.

Continue in the same manner, identifying the figure formed at the center each time: equilateral hexagon, equilateral octagon, equilateral decagon, equilateral and equiangular, therefore regular dodecagon. This is the first diaphragm. It is like the diaphragm of a camera. Bring one in to demonstrate.

2. Construct the second star. As before, divide into two equal parts. Slide one side so that the vertices of the extreme angles meet. Note the change from a point to a line segment. Continue naming each of the figures made, ending with the equilateral and equiangular, therefore - regular hexagon.

3. Construct the third star. Divide as before and slide one half. In this only the point, line segment and square are formed in the center. This is the third diaphragm.

Note: This third diaphragm will serve as a point of reference for two algebraic demonstrations of the Pythagorean theorem.

4. The children draw, cut, and paste the stars and diaphragms.

5. Older children may solve for the areas of the diaphragms and their internal figures, and find the relationship between them.

6. Constructing the second or third star, the child forms other figures by fitting in the angles.

7. Encourage further explorations using these triangles.

Direct Aim: Exploration of shapes using triangles.

Indirect Aim: Preparation for the sum of exterior and interior angles.

Chapter Two - Knowledge of Plane Figures and Details

Introduction

The two materials used in this chapter are the box of sticks and the classified nomenclature of geometry. The box of sticks is the most important instrument used by the teacher for the presentations, and in the succeeding work of the child.

The classified nomenclature codes the concepts given in the presentations. It is a bank of information to which the child will refer constantly.

Etymology continues to play a crucial role, as we take into consideration the different psychological realities of the child from 3 - 6 years old, and the child from 6 - 12 years old. From 3 - 6, the child has a drive to know "things", while from 6 - 12, the child has a drive to know "the reasons of things". Therefore at the level of language, the "thing" is given by its name, and the "reason of the thing" is given with the etymology of the name.

Materials:
Box of sticks
Accompanying box of supplies
Classified nomenclature
Wall chart of pictures and labels

Description of materials: Box of sticks
Eleven series of sticks, the first ten of ten different colors and ten different lengths; the last is a series of varied natural sticks.
dark brown - 2cm light brown - 12cm
violet - 4cm green - 14cm
orange - 6cm pink - 16cm (lengths are hole to hole)
red - 8cm blue - 18cm
black - 10cm yellow - 20cm
Eight of each of these ten series have holes at the extremities; two of each have holes all along the length. The natural sticks are of ten different lengths. These serve to construct right-angled isosceles triangles whose hypotenuse are equal to s 2. Therefore these sticks have the lengths of 2 2, 4 2, 6 2, 8 2 and so on.
Three groups of semi-circumferences and three circles.

red circle - diameter of 10; corresponding green semi-circumference
silver circle - diameter of 7; corresponding orange semi-circumference
white circle - diameter of 5; corresponding blue semi-circumference

Box of supplies:
four different colors of thumbtacks
red upholstery nails
brads
three crayons - red, blue, black

Classified Nomenclature for Geometry:

Series A is an exception since it also includes an envelope containing two white pieces of paper picturing a point and a line in red; a red square of paper (surface) and a cube constructed of red paper and dismantled to be stored in the envelope. Note: The line goes off the edges of paper to imply infinity.

Fundamental Concepts

While most secondary schools present these concepts starting with the point, the most abstract, and progressing to the solid, reality, we will in the elementary school begin with reality; the concept of the body and go on to the surface, line and point. In the second presentation after the child has worked with these concepts we will present them in reverse order: point, line, surface, solid.

Materials:
A box - parallelepiped, a can-cylinder, a ball - sphere
The Geometric solids and the stands
Classified nomenclature
Decimal system materials: cube, squares, bars and beads
Pencil sharpener, pencil, paper

Presentation:

A. The teacher asks the child to bring one of the three objects to place in the center of the table. Now put another object in the place of that one. It can't be bone unless the first object is moved. Try with the third object.
We see that we cannot place an object in a place occupied by another object. Everything that occupies a space is called a solid. This box has flat surfaces. The ball has curved surfaces, so it is the opposite. Since the can has flat surfaces and curved surfaces, it can be placed between them. Examine the geometric solids, naming them as they are divided into three groups: 1) cube, square-based parallelepiped, regular triangular prism, square-based pyramid, regular triangular pyramid; 2) cylinder, cone; 3) sphere, ovoid (or ovaloid), ellipsoid.

Touching the objects lightly, the teacher compares the surface to a very thin veil of paint, or to a piece of paper. With an inset or an object, the teacher lightly runs her fingers over the surface: this is the surface. We can touch anything in the environment and label it a surface; table, wall, face, the globe. The concept of surface is infinite. It goes on in all directions to infinity.

The teacher runs a finger along the edge of the box. This is a line, this is another line. continue identifying lines on other objects in the room, on the geometric solids. In reality the line as a concept has only one dimension, and it is infinite in both directions. Let's try to draw a line. Sharpen the pencil well. Invite the child to draw a line. That's too thick. Sharpen the pencil more and try again. This is okay, but it's still too thick. It should have no thickness at all. And it should be much longer. It should to out the door, across the fields, into the woods; and in the other direction as well, past the deck, through the field, down the street.

On the corners of the box, identify a point, another point. What is a point? It is nothing, but it is similar to a grain of sand, a particle of dust, a grain of pollen.

Let's try to draw one. Be sure the pencil has a sharp point. Invite the child to make a dot. That's too big, try again. It is still too big. The concept of a point is that element that has no dimension at all.

Note: Only the solid has a definition, the others can only be compared to things.

Bring out the special envelope for this series and construct the cardboard cube. The teacher identifies the solid, and the surface. Turn and bend the surface - it's still a surface. Identify the line and change its position - it's still a line. Identify the point. The child matches the labels and definitions using the wall chart and control booklet for control.

B. Note: The way in which the quantities of the decimal system were presented is very important now. The identification has already been made between the unit and the point, the bar and line, the square and surface and the cube and solid. The bead had to be held carefully for it is so small it might roll away. the bar gave an idea of length like a cane. The square covered the palm of your hand. The cube filled up the hand so there was room for nothing else.

The teacher places a unit bead on the table. This is a bead, a unit; it is a point. Add another point, still another point... I've made a line. The point is the constructor of the line.

Replacing the beads with a bar of ten, the teacher identifies the line. Other lines are added, making a column of bars. I've made a surface. The line is the constructor of the surface.

Replacing the bars with a square, the teacher identifies the surface. The squares are stacked; this is another surface, still another surface... I've made a solid. The solid is made up of many surfaces. The surface is the constructor of the solid. In reality, the point, therefore, is the constructor of the solid.

Imagine that this point (the bead) is on fire, like the end of an incense stick in a dark room or the phosphorescent plankton at night. Roll the bead. When it moves I see a line. This moving point has made a line before my eyes.
Imagine that this line (bead bar) is on fire. Hold it a one end and quickly move it back and forth. This moving line determines a surface.

Imagine that this surface is on fire. In moving the surfaces I create a solid.

The point is the constructor of reality. Take an object from the environment, a box or can. All things in reality are limited by surfaces. The surfaces, in turn, contain lines. the lines are made up of points, and the point is made up of nothing. Technically the point has no dimensions.

The same identification can be made of unit beads, when placed far away the cube itself appears to be no more than a unit bead.
The work with the classified nomenclature involves the picture cards in the folder, with reading labels and the same definitions.
Direct Aim: To furnish the fundamental concepts: point, line, surface, solid.

Indirect Aim: To furnish the concepts of all plane figures (which are simply surfaces) and all geometric solids (which are simply solids) in preparation for the study of area and volume.

Age: After 6 years

Note: Use the large dictionary which is gilt-edged; demonstrate sensorially the solid (the dictionary) is made up of many surfaces (pages). Isolate the gilt surface and show that it is made up of many line (edges of paper). One line is made up of many points (particles of gold powder).

Lines

A. Types of lines

Materials: A string attached to two small spools

Presentation: The teacher, unseen, places one spool in each hand and closes his fists so that the spools cannot be seen. The string passes between the fingers of each hand. Placing the fists together, the children are invited to watch. The teacher unrolls the spools as the hands are separated.. this is a line, this is a line... As the line grows, the teacher changes its position continuously - horizontal, vertical, oblique - still identifying it only as a line.
Finally, with the string taut, this is a straight line; and with it drooping, this is a curved line. "A line" cannot exist by itself; it must be straight or curved or a combination of both. Any line that I make will have one of these qualities. The concept of a line goes infinitely in both directions.

Exercises: Classified nomenclature and commands

B. Parts of a straight line

Materials:
...Two strings attached to spools
...Two scissors, a red felt pen

Presentation: Again the teacher prepares the spools in her hands and invites the child to watch as the line appears. What is it? A line...a straight line... a straight line.... Invite the child to find a point on this straight line and mark it in red. The child then takes the scissors to cut the line at the point... this is a straight line... (cut). Taking one spool and the point, the teacher extends the string... this is a ray... this is a ray. The teacher identifies the other ray in the same manner. These two rays are equal. a ray starts from a point and goes on to infinity. This red point is the origin. Therefore a point divides a straight line into two rays.

With the other spool the situation is repeated, with child identifying a straight line. This time the child is invited to make two red points on the line, and to cut the line at the two points simultaneously. Before the cut, the line is identified as a straight line. After the cut the teacher takes on piece at a time. This is a ray. This is another ray. this is a line segment. The two red points on this line segment are called endpoints, because we can't tell which is the beginning and which is the end. Therefore two points divide a line into two rays and a line segment.

Exercises: classified nomenclature - after the child has put the cards with the corresponding labels, have him put them in order: origin, ray, endpoints, line segment.

C. Positions of a straight line

Materials:
...A transparent pitcher and vase
...Water and red dye
...A level, two plumb lines, with a red line (cord)
...A red stick
...A globe

Presentation: Dye the water in the pitcher red (because red is always used to highlight the subject of a presentation) and pour some into the vase, which is placed in the center of the table. Observe the surface of this water and describe it: it is still. This will be a point of reference.

Agitate the pitcher and place it next to the vase. Let's wait without saying anything until the surface of this water becomes like the other. When it is exactly like the point of reference, the surface can be identified as horizontal. Place the red stick alongside the vase so that it aligns perfectly with the water. This stick represents a line which goes on in both directions. When it has the same position as the surface of the still water, it is a horizontal line.

Drop the stick in the water and wait until it is still (like the point of reference). This stick represents one of the lines that make up the surface of the water. This is a horizontal line. Remove the stick.

A straight line is horizontal when it follows the direction of still water (horizontal < horizon: Greek *horizon* < *horas*, boundary, limit; thus the horizon is the boundary of the visible earth in all directions, where it seems that the sky touches the water. Bring the children up the hill to see the horizon.

Hold the plumb line until it is still. This will be the point of reference now. Get another plumb line and wait without touching it, without a word, until it is exactly like the point of reference. Place the red stick along the red cord, so that it coincides just as the stick on the surface on the water. This straight line which goes on in both directions infinitely is vertical, because it follows the direction of the plumb line. This is a vertical line (vertical: Latin *verticalis* < *vertex*, whirlpool, vortex, crown of the head, summit, highest point, <*vertere*, to turn; therefore vertex can be applied to anything which turns like a whirlpool, or to the highest point, like the crown of the head or the summit). A vertical line is one which points to the vertex, that is, the topmost point in the sky over our heads (zenith). It passes through the center of the earth and on to the nadir (opposite of zenith). Take the plumb line and hold it still again. Let's imagine that this is a straight line which goes on in both directions - up to the zenith and down through the center of the earth to the other side. Use the globe to show that a vertical line is relative to the position of the observer.

Place the points of reference for the two opposite elements in front on the child. What is the median? Hold the red stick horizontally. When a straight line follows the direction of the surface of still water, what is it? A horizontal line. Hold the stick vertically. When a straight line follows the direction of the plumb line what is it? A vertical line. Hold the stick obliquely. Is this straight line like the surface of water? The plumb line?

When a straight line is neither horizontal or vertical, it is oblique. Turn the stick 360o identifying its position as it turns - horizontal, oblique, oblique, oblique....vertical, oblique, oblique, oblique, oblique, horizontal, oblique ...etc. (oblique: Latin *obliquus*, slanting, sloping, not straight, not right, devious) So what is straight, right, and normal? The horizontal and vertical line. The oblique line runs contrary to the true, contrary to vertical or horizontal.

Exercises: Classified nomenclature and commands. Demonstrate use of the level for determining lines in the environment.

D. Horizontal line - curved or straight?

Materials:
...A globe
...Frame of smallest circle inset
...Knitting needle

Presentation: Invite the child to identify their hometown on the globe. Place the inset frame on the globe so that the town coincides with the center of the circle. This curve (the rim of the frame) represents the horizon for everyone living here in and around the town. Because we are standing outside the earth we see the entire horizon as a circle, instead of as an arc-part of the circumference.
With the chalk mark a point on the floor. Draw a circle around this center point. Invite the child to stand at the center. What do you see? Without turning the child can only see an arc, a part of the circumference.

Reinforce the facts that in these demonstrations the child is much bigger than the circle on the globe or on the floor, when in reality it is the reverse. The child is a tiny, tiny point in relation to the earth which is huge. The curvature of that arc would be so slight that you would only be able to see a straight line.

With the knitting needle, hold it so that it forms an arc on a horizontal plane. Invite the child to identify what he sees as he lowers his body..... a curved line.... a curved line... a straight line. At eye level this curved line looks like a straight line.

E. Straight line in a horizontal plane

Materials: Box of sticks, supplies, board

Presentation: The teacher tacks a stick onto the board and identifies the board as a plane; a straight flat surface that continues in all directions infinitely. Imagine that this stick is a straight line that goes on infinitely in both directions.
Hold the plane vertically and ask the child to identify the position of the line as the plane is rotated: 1800...horizontal..oblique...oblique...vertical...oblique...etc.

Hold the plane obliquely. The plane in space could be in any position, but to facilitate your work, the plane will always be horizontal, like the surface of your work table.

Tack on two other sticks so that the three positions are represented: -, /, |. If these three lines were considered in space they would all be horizontal (hold the plane at eye level to show this). Let's consider them on the plane surface. When a straight line follows the direction of the viewer's body, it's vertical. When a straight line doesn't form a cross or follow the same direction of the viewer's body, it's oblique.

Exercises:
1. Leave only one stick on the board. As the plane is rotated (always horizontally) the child identifies the position of the line.
2. To understand that these positions are relative to the viewer, seat two children so that a right angle is formed between their bodies and the plane. As the plane is rotated, the children simultaneously identify the line as they view it.
3. The child draws the lines on the blackboard. The criteria hold true even if the plane is vertical (or oblique) like the blackboard.

F. Two straight lines lying in a plane - coplanar lines

1. Parallel lines

Materials:
...Box of sticks, supplies, board
...Figures of children
...Red arrows

First Presentation: Identify the board as a plane. Place a stick on the board and identify it as a line belonging to the plane. We've already explored everything we can about this straight line. Let's see what happens when we add another straight line. Put another stick on the board so that it neither touches or crosses the first. We have two coplanar straight lines; they both belong to the same plane (coplanar: Latin *con*, together, *planus*, plane; thus lying in the same plane).

One stick is fixed to the board horizontally. Taking two small equal sticks, these are the key to the story. Place them perpendicularly along one side of the first stick. Move the second stick toward the first until it meets the guide sticks and fix it there. Remove the guide sticks, but leave them nearby to remind the child of their importance.

Place the two indifferent children on either sides of the lines so that they are walking in the same direction. The expressions on their faces show indifference. It is as though they don't even know each other. Move the figures along to the end of the line and turn them over; make them walk back. They are like two people walking on opposite sides of the street. They don't care to know each other. each one stays on his own sidewalk and they will never have the chance to meet.

We can extend these straight lines to infinity (add sticks of the same color, fixing them with the guide sticks until the lines go off the board in both directions) but these two lines will never meet. Substitute the red arrows for the two children. These two lines are parallel. They never meet no matter how far we follow them because they are always the same distance apart (parallel: Greek *parallelos* < *para*, beside, and *allelon*, of one another; thus one thing beside another)

Exercises: find parallel lines in the environment - door frames, fence rails, telephone wires, rows in the garden, railroad tracks.

Second Presentation (Parallel lines are parallel independently of their position): Invite the child to construct two parallel lines and then to identify their position: horizontal. Ask the child to construct two vertical parallel lines and then two oblique parallel lines using the same process as before. All are parallel regardless of their position.

Remove two pairs of parallel lines. Rotate the plane in its horizontal position as the child identifies the position...horizontal...oblique....vertical...etc.

Whenever we draw two parallel lines, the lines are also horizontal, vertical, or oblique.

Construct a series of parallel lines, using the same guide sticks or a pair of longer guide sticks. These are called "fascial lines" because this was the symbol of fascism, first used be Julius Caesar and later by Benito Mussolini.

G. Divergent and Convergent Lines

Materials: Box of sticks, supplies, board
Also 4 figures of children: 2 happy, two sad and 4 one-way red arrows

Presentation: The two parallel sticks may be left on the board for comparison. The teacher fixes one stick horizontally. Two small, but different guide sticks are used to position the second stick. The guide sticks are set aside.

Place the two unhappy children on the lines. These two children are very sad. They used to get along very well, but as they went along in life, the distance between them becomes greater and greater. (move the figures along the lines) That's why they look so unhappy.
Replace the figures with one way arrows. These lines go only in one direction, the distance between the lines keeps increasing. Place extra sticks at the wide end, showing that the guide sticks would also need to increase in length. These are divergent lines.
(divergent < diverge: Latin *di*-apart, *separatelym* and *vergo* - to incline 0r -
Latin *divergare* < *devergere, de*-opposite of *con* (together) and *vergare*, to direct oneself; thus to move away from each other). This term was coined in 1611 by Kepler to give the opposite of convergere which means to direct towards each other.

Exercise: Find divergent lines in the environment

Presentation: Position one stick horizontally on the board. As in the preceding presentation use two different guide sticks to position the second stick. Fix the second stick and set the guide sticks aside.

Place the two happy children at the wide end. As these two go along, they become closer and closer and happier and happier, knowing that in the end they will meet.

Replace the figures with one-way arrows. These lines go only in one direction - toward each other. These are convergent lines. (convergent: Latin *con-* together, and *vergera* - to incline) These lines come from two different points toward each other to one point.
Love stories in geometry, like those in real life can change. Place extra sticks at the narrow end to see how these line continue in their one direction. What happens? After the point of convergency, these two lines become divergent.

Exercise: Find convergent lines in the environment

H. Oblique and perpendicular lines

Materials:
...Box of sticks, supplies, board
...Measuring angle

Presentation: Take two pairs of sticks with holes along the length and connect each pair with a brad at the center. Let's see how two straight lines can meet. Two straight lines can meet this way X or (rotate the second pair from an overlapping position, through the position shown so that the child may see that they are equal and then on to a perpendicular position) two straight lines can meet this way + (Note: each pair started form a horizontal position).

Invite the child to measure the four angles of the first pair to see if they are right angles. None are right angles. In the second pair, all are right angles.

When two straight lines meet and do not form angles that are right angles, the tow straight lines are oblique to each other. Review the meaning of oblique (deviated, slanting, not right).

When two straight lines meet and form all right angles they are perpendicular to each other (perpendicular: Latin *perpendicularis* < *perpendiculum*, a plumb line < *per*, through and *pendere*, to hang). this perpendicular line hangs and goes through the other Note: the Old English word for plumb line is perpendicle.

Three period lesson with child constructing them.

Exercises:
1. Place a pair of overlapping sticks horizontally with the measuring angle positioned at the vertex. Ask the child to identify how the lines are in relation to one another as the top stick rotates ... oblique, oblique... perpendicular, oblique .... as they overlap again - silence) .... oblique ... etc.
2. the child is asked to take three pairs of sticks and unite them with brads in this way:
1st pair - both have hole along the length; united at the center
2nd pair - one has holes, the other is normal; united at the center of the one with holes
3rd pair - both have only end holes; united at one end.
The sticks are lain overlapping in horizontal positions. Using the measuring angle the child makes the first pair perpendicular and counts the right angles formed (4). The number is written on a piece of paper and is placed by the pair. The same procedure is followed for the second and third pairs. When two lines meet and are perpendicular to each other, they create four right angles, or two right angles or one right angle. Invite the child to try o arrange two perpendicular lines that create three right angles. It is not possible.
The first pair are two straight lines; the second are a line and a ray; the third are two rays.
3. With one pair of sticks with holes joined at the center and placed horizontally on the board, the child is asked to make the two line perpendicular, checking with the measuring angle. These lines are

perpendicular. The teacher turns the whole thing 450 and measures the angles to check. How are these lines in relation to each other? Still perpendicular. Before the lines were horizontal and vertical, now both are in an oblique position. Do the same with the second and third pairs from the previous exercises. With the measuring angle, show that right angles are always formed, regardless of the position of the lines. Therefore all of these lines are still perpendicular to each other because the amplitude of the angle didn't change.

I. Two straight lines crossed by a transversal

Materials:
...Box of sticks, supplies
...Board covered with paper

Presentation: Place one, then another like stick on the board, having the child identify the number of straight lines on the plane. Then place a third stick (a different color with holes along the length) so that it crosses the other two. Now there are three straight lines on our plane; the third crosses the other two.

Remove the sticks. Place one horizontally and tack it down reminding the child that this straight line goes on in both directions to infinity. Into how many parts does it divide the plane? Indicate these two parts with a sweeping hand. Place the second stick on the plane so that it is not parallel. Even this straight line goes on to infinity. With a black crayon, draw lines to demonstrate this. Identify the three parts into which the plane has been divided. The part of the plane which is enclosed by the two straight lines is called the internal part which we can shade in red. Above and below the straight lines are the external parts of the plane because they are not enclosed by these two lines.

Place the third stick across the other two and tack it down where it intersects. This is a transversal (transversal < transverse: Latin *trans*, across, and *versus*, turned; thus lying crosswise). Two straight lines cut buy a transversal on a plane will determine a certain number of angles - how many? Using non-red or non-blue tacks, identify and count the angles. First conclusion: Two straight lines cut by a transversal will form eight angles.

Some of these angles are lying in the internal part of the plane, while others are lying in the external part. Remove the tacks. Identify and count the angles in the internal part, using red tacks (same color as the plane). These four angles are interior angles because they lie in the internal part of the plane. Do the same, identifying the exterior angles. The four angles are exterior angles because they lie in the external part of the plane. Second presentation: Two straight lines cut by a transversal form four interior and four exterior angles.

We need to divide these eight angles according to different criteria. Remove the red and blue tacks and identify two new groups using two other colors: four angles formed by one straight line and a transversal; and four angles formed by the other straight line and a transversal. All of the work that we'll be doing involves pairing an angle from one group with an angle from another group. We won't be working with two angles from the same group because that would mean only two straight lines were being considered, not three. Let's examine these pairs.

Remove the tacks. Using two tacks of the same color, the teacher identifies two angles. These two angles are a pair of alternate angles. Recall the meaning of alternate. One is on one side; the other is on the the other side of the transversal. On what part of the plane are they? Internal, therefore they are also interior angles. We combine these two characteristics into one name: alternate interior angles. Invite the child to identify the other pair with two tacks of a different color. The child draws these and labels them.
Remove the tacks. The teacher identifies another pair of angles. These are a pair of angles that lie on the same side of the transversal. On what part of the plane do they lie? Internal, therefore they are also interior angles. We can call these interior angles that lie on the same side of the transversal. Invite the child to identify another pair with two tacks of a different color. The child draws the angles

and labels them appropriately.

Remove the tacks. The teacher identifies another pair of angles. These are alternate angles because they lie on on one side one on the other side of the transversal. The child identifies in what part of the plane they lie - external - and their corresponding name - exterior. These are alternate exterior angles. Invite the child to look for another pair and identify them with two tacks of a different color. The child draws the situation and labels it accordingly.

Remove the tacks. The teacher identifies two angles. These are a pair of angles that lie on the same side of the transversal. The child identifies in which part of the plane they lie - external - and recalls their subsequent name - exterior. Therefore these angles can be called exterior angles that lie on the same side of the transversal. The child is invited to identify another pair using two tacks of a different color. The child copies this situation and labels it.

Remove the tacks. This time an exterior angle will be paired in a relationship with an interior angle. The child chooses an angle, identifying it with a tack. The other angle must be formed by the other straight line, as you remember, so that three lines will be involved. The teacher identifies the other angle of the pair. These are corresponding angles, because they follow a certain order. Both angles lie on the same side of the transversal, and each angle lies above its straight line. Invite the child to identify other pairs using different color tacks for each pair of angles. All eight angles are used. The child copies the situation and labels it accordingly. Note: These angles have only one quality, since the pair is divided among the two different parts of the plane.

Finish with classified nomenclature and commands. A command might ask the child to identify the other member of a given pair of angles.

J. Two parallel straight lines crossed by a transversal

Materials:
...Box of sticks, supplies, board
...Twelve xeroxed forms showing above
...Paper, pencil, and ruler

Presentation: With the sheet of paper from the board and the sticks arranged as they were for the last series of presentations, nearby, construct two parallel (horizontal) lines crossed by a transversal. Recall construction of parallel lines from before using the guide sticks. Position the transversal and fix it with tacks.

Verify that the nomenclature used in the last situation is also applicable here: internal and external parts of the plane (red and blue shading is no longer necessary); interior and exterior angles, and so on, until the four pairs of corresponding angles have been identified. Since the same nomenclature is used, and the same angles exist, the old plane and its lines can be removed. What we must discover now is how the angles which constitute a pair are related.

Invite the child to choose a pair of alternate angles, identifying them with a pair of tacks which are the same color. The child names the pair he has chosen: either alternate interior or alternate exterior angles. The child then colors these angles on one of the forms. Note: This is only a sensorial demonstration of congruency.

Cut the form along the transversal. This will always be the first of the two cuts we must make. Choose one of the two resulting parts and identify the sides of the angle to isolated. Cut the form along the side of the angle. Superimpose this angle over the other angle, sliding it along one side until the vertices and other sides meet, showing that the two angles are congruent. Repeat the procedure with the other pair of alternate exterior angles, and the two pairs of alternate interior angles. Upon completion, it will be evident that all possibilities for alternate angles are exhausted, because the angles on the board will each have a tack.

Use a new form for each pair of angles considered. After each pair, make a conclusion about the congruency of that pair of angles. At the end we can make this generalization: There are four pairs of alternate angles. The angles of each pair are congruent to each other.
Invite the child to choose a pair of corresponding angles, identifying them with a pair of tacks. Remember that corresponding angles are not differentiated by the name interior and exterior, because one of the pair is lying in the internal part of the plane while the other is in the external part of the plane. The teacher controls the choice of angles; both angles lie on the same side of the transversal; both have the same position with respect to their line; i.e. they are both above their lines. The child transfers this situation to a form by shading in the angles.

As before, make the first cut along the transversal. Notice that this time both angles are still on the same part of the form. Make the second cut along the line that will divide the two angles, so that we will be able to superimpose them. Superimpose the angles as before to demonstrate congruency. We can conclude that: The first pair of corresponding angles are congruent.

Identify the second, third, and fourth pairs of corresponding angles. Follow the same procedure, using a new form each time, to demonstrate that the angles are congruent. Again, in the end the angles will all be identified with tacks. Generalization: There are four pairs of corresponding angles. The angles of each pair are congruent to each other.

This time, instead of demonstrating that a pair of angles are congruent, we must demonstrate that they are supplementary. This time we will again have two pairs of interior angles and two pairs of exterior angles. Invite the child to identify two angles that lie on the same side of the transversal with a pair of tacks. The child names the pair which he has chosen: interior or exterior angles; and transfers them to a form. As before cut along the transversal to find that both angles are still on the same piece of paper. We must separate the angles by the second cut. Place the two angles side by side so that their transversal sides are adjacent and their non-adjacent sides form a straight line; thus demonstrating that the two angles are supplementary. We can conclude that the first pair of (i.e.) interior angles that lie on the same side of the transversal are supplementary.

As before, repeat for the second, third, and fourth pair, making a conclusion with each pair. In the end, all of the angles have been identified with tacks. We can make this generalization: These are four pairs of angles which lie on the same side of the transversal. The angles of each pair are supplementary to each other.

Exercises: The teacher prepares forms with only the parallel lines. The child in his work completes the form by drawing the transversal either way. The direction of the transversal cannot be changed by changing the position of the form.

1. The child uses the forms to demonstrate that each pair of angles is congruent or supplementary. For each pair he writes the appropriate conclusion. At the end, he writes the generalization (as in the presentation).

2. The child works from command cards. For example: Demonstrate that the angles that constitute any pair of alternate angles are congruent. Note: In order to do this, the child must have realized that the two straight lines cut by the transversal must be parallel. He may choose interior or exterior angles.

Age: After 9 years

Direct Aim: Knowledge of the theorems involved regarding congruent angles and supplementary angles formed by two straight parallel lines and a transversal

Indirect Aim: Preparation for a more advanced study of parallelograms

Angles

A.Types of Angles and the Parts of an Angle

Materials:
...Box of sticks, supplies, board
...Red felt pen (with a long thin head)
...The measuring angle, in its envelope (both sides are colored)

Presentation: The teacher takes two different sticks - a long one with holes along its length and a shorter one with just two end holes. Placing the longer one on top of the other at the center of the board the two sticks are fixed at the lower end with a red upholstery tack. At this point both are mobile, so place the shorter one in its vertical position and fix it there. Now only one moves.

Place the red pen in the last hole of the top stick and begin drawing a line in a clockwise direction... this is an angle, this is an angle.... (indicate the angle with one hand moving around between the sides of the angle) ... (at 3600) this is a whole angle. It is called a whole angle because I went all the way around to the same point again.

Watch where I stop this time. Repeat the same procedure. This is an angle, this is an angle... (at 1800) this is a straight angle. The line it has made is a straight line.

Repeat the procedure as before, starting from zero. This is an angle, this is an angle ... (at 900) this is a right angle. Take out the measuring angle and identify the red part which is the "measuring angle". Note: this is the child's first protractor). The measuring angle is a right angle. Demonstrate that the angle just constructed is truly a right angle by sliding the measuring angle along one stick until it meets the other line... a perfect match.

Leave the measuring angle there. Repeat the procedure as before, starting from zero. This is an angle, this is an angle ... (before 900) this is an acute angle. It is less than the measuring angle.

Leave the measuring angle in its place. Repeat the procedure as before, starting from zero. This is an angle, this is an angle ... (after 900) this is an obtuse angle. It is more than the measuring angle.

This red nail is the vertex of the angle. These two sticks are the sides. They are rays which continue to infinity in one direction. The inside part of the angle is the size of amplitude.

Using the measuring angle the child measures the size of each of the angles constructed. The acute angle is less than the measuring angle; the right angle equals the measuring angle; the obtuse angle is more; the straight angle is twice the measuring angle; the whole angle is four times the measuring angle.

To demonstrate the theorem that the size of an angle does not vary with the length of its sides, place the pen in the second hole and draw a right angle again. Measure it to see that it is the same.

Exercises:
1. classified nomenclature
2. Draw angles and cut them out. Classify them using the measuring angle.
3. Look for the various types of angles in the environment. The child will notice that most angles are right angles.

Age: Seven years

Aim: To give the first concept of angle, to explore the different types of angles, and to give nomenclature of the types and parts of an angle.

B. Study of angles

Materials: Box of sticks, measuring angle

Presentation: Take four sticks, of which two are alike. Separate the four into pairs having one each of the like sticks. Join the pairs with brads to form two angles in such a way that one of the like sticks is on top and one is on the bottom.

These are two different angles; they have nothing in common., Place the angles so that the like strips are superimposed. Now these two angles have one side in common. To emphasize that these two angles have three sides, disjoin the angles, remove one of the like sticks and rejoin the three with one brad. There are two angles and three sides; they have one side in common, and therefore a common vertex.

The two sides which are not in common are not opposite rays (opposite rays form a straight angle).

These angles are called adjacent (consecutive) angles (adjacent: Latin *ad*, to, near and *jacere*, to lie; thus to lie near each other) (consecutive: Latin *consequa*, one thing that comes after another). Put these angles aside, intact for later use.

Take four sticks, two of which are alike. As before, form two angles so that one like stick is on top; the other is on the bottom. These are two angles; there are two vertices and each angle has two sides. Superimpose the like sticks, then separate the sticks and remove one like stick, then rejoin the three. Now there are two angles, but they have one side in common and thus a common vertex. The two sides which are not in common are opposite rays. These are adjacent angles. Put them aside for later use.

Take four different sticks and connect them to form two different angles. The teacher then places them, side by side but not overlapping. There are two angles, two vertices, and each angle has its own two sides. They have one special characteristic (place the measuring angle under the sticks to demonstrate), the sum of their angles is equal to the measuring angle. These two angles are complementary (complementary: Latin *complementum*, that which completes or fills up the other; thus they complete a right angle).

Take four different sticks and join them to form two different angles. Position the angles. there are two angles, each having their own vertex and two sides. However there is one special characteristic. Their sides form a straight angle. The sum of their angles is 1800. Place the measuring angle to show that the angle formed is the double of the measuring angle, and therefore it is a straight angle.these are supplementary angles (supplementary: Latin *supplea*, to fill up, more than). The etymology doesn't help much. The angle formed is "more than" the measuring angle, in fact it is the double.

Take the sample of complementary angles made previously. The child identifies them and recalls their characteristics - sum is equal to 90o. Take also the adjacent (consecutive) angles. The child identifies them and recalls their principal characteristic - one side in common.

Take four sticks, two which are alike and join them to form two angles as before. Slide the two towards each other to make a 90 angle and verify this with the measuring angle. The child identifies these angles: only complementary. Disconnect them, remove one like stick and rejoin them. Position them again and verify with the measuring angle. The child identifies them again as having both characteristics: adjacent and complementary. These two angles are complementary adjacent angles.

Take four sticks, two of which are alike. On the board, the supplementary angles and adjacent (case 2) angle previously made, have been placed for reference. The child identifies them and recalls their principal characteristics.

As usual the four new sticks are joined to form two angles: one like stick on top; one like stick o the bottom. These two angles are only supplementary. Remove one like stick and rejoin the angles. These two angles are supplementary (because their sum is a straight angle) and adjacent (because they have one side in common) They are supplementary adjacent angles.

Finish with a three period lesson. The child works with classified nomenclature. They may draw, cut, paste and label the angles.

C. Vertical or Opposite Angles

Materials: Box of sticks, supplies

Presentation: The child brings four like sticks and constructs two angles connecting them with brads. These two angles have nothing in common. Let's give them a common vertex. (simply superimpose them without disjoining and rejoining them). Now let's move the sides of one angle. This side is the opposite ray to this side. In the same way indicate that the other two are opposite rays. Since this set-up isn't too steady, we can reconstruct them using two sticks with holes along their lengths. Join them with a brad somewhere near the center and tack two of the ends down. (note: they are oblique lines - not perpendicular)

Identify the four angles that have been formed using 4 tacks of the same color from the box. This angle is on the opposite side of the vertex from this angle. These two angles have a common vertex. In the same way identify the other pair of angles as being opposite the vertex from one another. The characteristics of opposite angles have been identified: they have a common vertex and their sides are opposite rays.

Note: At three different age levels, there are three different demonstrations to show the equality of opposite angles:

1. Sensorial demonstration (7 1/2 years) On a piece of paper repeat the situation of the sticks, using these to trace two lines in black which go off the page. This line represents one stick; this line represents the other. Note the two angles indicated by red thumbtacks and color them with crayon on the paper. Do the same for the other two angles. Cut our plane (the paper) along one of the black lines, thus dividing the plane into two semi-planes. Take one of these and cut along the ray, dividing the plane into three pieces.

We must satisfy that this (loose) angle, which is colored red is equal to its opposite angle, shaded in the same color. Superimpose the loose angle at the edge, matching the sides. Slide the piece along that side until everything meets perfectly. In the same way demonstrate that the other pair is equal.

2. (approx. 8 1/2 years) Take the envelope entitled "vertical angles" from the box of supplies which contain four cards on which 1 - 4 are written. Remove the tacks and number the angles. We must show that angle 1 = angle 3, and angle 2 = angle 4. When writing we can use this notation ^1 to say "angle 1". If we add ^1 and ^2, since they are adjacent and supplementary they will total 1800 (if the child has not learned how to use a protractor we say ^1 + ^2 = 2 measuring angles). Likewise, ^2 and ^3 form a straight angle. Placing a straight edge along the sticks isolates this characteristic, making it more visible to the child. Indicate the angles emphasizing the common angle - the common addend - angle 2. There for ^1 = ^3. Continue in the same way.

3. (approx. 11 years) given a diagram of the angles, we mark them with an arc to show they are equal; a double arc for the second pair. Simply state the textbook declaration: Angles 1 and 3 are both adjacent to angle 2. These are supplementary to the same angle, angle 2. therefore angle 1 and angle 3 are equal to each other. Finish the lesson with classified nomenclature and drawing of the angles.

D. Measurement of Angles

Introductory Presentation:
Note: Up until this point the child has only measured angles with the measuring angle, and before that with reference to the right angle (in the geometry cabinet drawer of triangles). Here we will introduce the concept of degrees.

When we measure things we use certain units of measurement. For water, we measure the length of an object in inches and feet. In other parts of the world they use liters, grams and meters. But when it comes to measuring angles there is a universal system that was invented a long time ago by Babylonian priests.

"The Story of the Star"
In ancient times there were some Babylonian priests who were very interested in astronomy and the calendar. These were priests who were accustomed to sleeping during the day and staying awake at night to look at the sky. They studied the paths of stars, and the constellations.

They discovered that in order to see a star in the exact same place as before, a certain number of days would pass. They counted and counted again, and by trial and error they arrived at the conclusion that 360 days would pass from the time a star was visible in a certain place until it returned to that same place again. They called this period of time a year.

They made a slight error in their calculations; they were off by 5 days, 5 hours and 49 minutes. But considering their instruments that they used for measuring, this wasn't much of a mistake.

This time period of a year was too large to be practical; they needed a smaller space of time. They counted the dawns from the appearance of this star to its reappearance and divided the year into 360 days.

The priests thought that the paths followed by the stars made a circle. So they divided the circle into 360 parts and called each part a day. We call these parts degrees.

From that day onward no one has ever changed that measurement, even though we have found out with our more modern instruments that they had made a slight mistake.

From this path of the star in the sky, the Babylonian system of numeration was developed. Their system of numeration was based on 60. The priests discovered that the circle could be divided into six parts. A regular hexagon could be inscribed in this circle, then divided into six equilateral triangles(where two sides are formed by radii, and the third by a cor equal to the radius). Taking 1/6 of the path of the star (circumference), they obtained 60 which is the base of their system.

The Babylonians also gave us their symbol for days which is our symbol for degrees. Instead of writing the word degree(s) after a number we simply use this symbol 0. A little circle to remind us of the path of a star in the sky. Now instead of talking about angles in terms of their size: being wide, big or small, we will talk of the amplitude of an angle.

Materials:
...Montessori protractor and other protractors
...Circle, square and triangle fraction insets
...Ruler and compass

1. Presentation of the Montessori protractor
This is the instrument used to measure angles. Its rim, like the path of a Babylonian star, is a circle. The Babylonians had determined that it took 360 days for the star to go around and come back to the same point; therefore we have divided this circle with little lines into 3600 (degrees).

With the child start at zero, where the star started, and count by ones up to 20, by 20's around to 340 and by ones to 360. But 360 is not written there, because we have reached the same place from where we started- zero. There's a line that runs from zero to the center of the circle. The center represents the point where the priests were standing in order to see this circle in the sky.

2. Use of the Protractor

Bring out the circle fraction insets of the thirds, ninths, sixths and halves. Taking 1/3, identify the angle to be measured: the only true angle on the piece. Recall the nomenclature and identify each part of the angle: angle, vertex, 2 sides.

Holding the knob of the piece, place the vertex on the red dot which is the center. Place one side down along the black line so that the side touches zero. Then place the inset piece flat, so that the side will touch one of the degrees. From zero count by 20's around to the other side - 120. Therefore one third is 120 degrees. The child writes 1/3 = 1200.

Try with 1/9 placing the inset piece on the protractor as before - verte, side along the black line, surface 1/9 = 400 Go on to 1/6 : 1/6 = 600. This is the subdivision that established their system of numeration. The star had followed its path for 60 days of the 360 days which is the whole angle.

With 1/2, identify for the child the angle and the vertex, and the two sides of the angles. The vertex cannot be placed at the center (in the priests room) as before. However the shape helps to line up the sides. We already know that this is a straight angle. 1/2 = 1800
Examine a second group of fraction. Bring out the fourths, eigths, fifths, and tenths. For each of these the second side meets a little line that is not numbered. Following the circumference of the circle from zero, count by 20's then by 10's, 5's or 1's to reach the second side.

The last group consists of the whole and the sevenths.

Remove the whole from the frame. Identify the angle, which is all of the interior, the vertex and the sides, which extend from the vertex in all directions. When this inset is placed into the frame, we cannot determine where the angle begins or end. The important thing to note is that everything is covered. Therefore the unit is 3600 - the whole angle. It follows the complete path of the star.
In measuring 1/7, we find that the side does not meet one exact mark. We can say that 1/7 is approximately 510.

Examine the triangle fraction insets. Begin with the whole triangle. Choose one angle, position its vertex on the center, one side on zero and read the measure on the other side. Continue for the other two angles. All angles are 600 therefore this equilateral triangle is equiangular as well.

The 1/2 piece has angles of 300, 600, 900. 1/3 has 1200, 300, 300. 1/4 has all 600 angles, just like the whole.

Examine the square fractions insets which are triangles formed by subdividing the square: 1/2, 1/4, 1/8, 1/16. All of the triangles have angles of 450, 450, 900. With 1/16 use a ruler to prolong the side,

Examine the rectangles and squares formed by subdividing the square: 1, 1/2, 1/4, 1/8, 1/16. All have all 900 angles. Again a ruler must be used to extend the sides of 1/8 and 1/16.

Ages: around 8 years

Aims: Measurement of angles
Indirect preparation for the sum of interior and exterior angles of polygons

Operations with Angles

a. Addition

Materials:
...Circle fraction insets
...Montessori protractors

Presentation: The teacher proposes an example: 1/2 + 1/4. The child isolates the two pieces, measures them one at a time and notes their measurements 1/2 = 1800, 1/4 = 900. We could restate the addition in terms of degrees: 1800 + 900. Place the 1/2 piece in the frame of the protractors and add the 1/4 piece. The result can be read on the protractor where the non-tangent side meets the frame. 1800 + 900 = 2700.

b. Subtraction

Materials:
...Circle fraction insets
...Montessori protractors

Presentation: The teacher proposes an example 1/3 - 1/6. The child isolates the two pieces, measures them one at a time and notes their measurements 1/3 = 1200, 1/6 = 600. Place the 1/3 piece in the protractor. From this we must take away 1/6.

1st method: Slide the 1/3 piece counter-clockwise the number of degrees corresponding to 1/6. Read the result where the second side of the angle meets the frame - 600

2nd method: Place the two pieces in the frame - minuend first. Slide the two pieces counter-clockwise until the non-tangent side of the second angle meets the measurement of the first angle - 1200. Take away 1/6. Read the result where the second side of the angle meets the frame - 600.

c. Multiplication

Materials:
...Circle fraction insets
...Montessori protractor

Presentation: The teacher proposes an example: 1/10 x 8. The child isolates the piece 1/10, measures it and notes its measurement. 360 x 8. Take the fraction 8 times placing them in the frame starting at zero and reading the result where the second side of the last fraction meets the frame.

Another example is 1/10 x 12 which is 360 x 12. Place all of the 10 pieces on; the angle is 3600. Remove them and place only two pieces on the protractor; the angle is 720. Add 3600 + 720 to obtain the results. In terms of fractions rather than angles, this result would be a mixed number.

d. Division - Bisecting an Angle

Materials:
...Full sheets of paper
...Compass, ruler

Exercise
1) An angle can be divided into two equal parts by folding the paper so that the two sides of the

angle (paper) meet. Recall the nomenclature of an angle. This fold is called the bisector, it divides the angle into two equal parts.
2) Draw an angle. With a compass, place the point on the vertex and mark off two equidistant points on the sides. Open the compass wider and with the compass point on one side point, then the other, draw two arcs. A line drawn from the vertex to the intersection of these two arcs is the angle bisector. Therefore the two angles are equal.

Aim: Preparation for geometric constructions

e. Exercises in measurement and drawing of angles - other protractors

Materials: Various protractors - circular protractors (small enough to fit in Montessori protractor), those having clockwise numeration, counter-clockwise; also semi-circular protractors of various sizes

Presentation: Introduce the other protractors, comparing the first to the Montessori protractor by placing it inside the frame and aligning the degree marks. Show the others, reinforcing the theorem that the amplitude of the angle does not vary with the length of its sides.

Demonstrate the use of the protractor in measuring angles. Place the center hole over the vertex of the angle. Align one of the sides to zero, by sliding the compass around like a wheel. From zero, read the numeration progressively until the place where the second side corresponds to a degree mark. The child writes the angle measurement inside the angle, with degree symbol in red. the teacher can prepare many angles out of cardboard which the child traces and measures.

Demonstrate how to draw an angle. Make a point in red. this will be the vertex. From this point draw a ray; this will be one of the sides. Place the protractor so that the vertex corresponds to the center hole, and the side corresponds to zero. Make a mark at the number of degrees desired. Draw a line from the vertex through this mark to make the second side. Write its measure inside. The teacher can prepare command cards which tell the child to construct an angle of a stated number of degrees.

Age: About 8 years

Aim: Use of standard protractors, measuring and constructing angles, and operations with angles, including constructing a bisector

E. Convex and reflex angles

Materials: Box of sticks, supplies, board
Red pen (with a long narrow head)

Note: Up to this point an angle has been defined: "that part of a plane lying between two rays which have the same origin", and five angles were identified: acute, right, obtuse, straight and whole.

Presentation: (By size) Invite the child to choose two sticks and fix them on the board as usual for making angles (the longer stick on top of the shorter and secured at one common end with the upholstery tack; the shorter is fixed at the other end as well). This pair of sticks should be toward the left side of the board, to leave room for the next pair.

Ask the child to make an acute angle or a right angle or an obtuse angle using the red pen. Identify the angle that is constructed. This angle is a convex angle. Color the angle. Because the child was given the choice of three angles, we can say that all three angles - acute, right, or obtuse are convex angles.

Choose two sticks identical to the first pair and fix them on the board. This time draw an angle which is greater than a straight angle, but less than a whole angle. This is a reflex angle. Color the angle with a different color (convex: Latin *convexu*s, rounded or bent, curved, crooked) (reflex: Latin *reflexu*s, bending back). Do a three-period lesson.

Presentation: (By sides): Begin from where the last lesson left off. Recall the nomenclature of the convex angle: vertex, side, side. ask the child to take two sticks just like the ones on the board and place them so that the sides of the angles are extended (prolonged). The existing sticks represent rays, therefore they extend the sides in the opposite direction. Place the loose sticks in position without fixing them. This green stick is a prolongation of this side of the angle, etc.., for the reflex angle as well.

Consider the first angle. Indicate the colored region. What type of angle is this? Convex. Do the prolongations of the sides fall inside this angle? No. We can conclude that the prolongations of the sides of a convex angle do not lie within the angle itself.

Consider the second angle with the same questions. Conclude that the reflex angle does contain the prolongation of the sides. Do a three-period lesson.

(Organization of the definitions) Combine the two viewpoints of size and sides. An angle whose amplitude is less than that of a straight angle is a convex angle. It does not contain the prolongation of its sides. An angle whose amplitude is less than a whole angle but greater than a straight angle is a reflex angle. It does contain the prolongation of its sides. .

Classified nomenclature

Exercise: With only one pair of sticks fixed on a clean surface, slowly construct the spectrum of all of the angles while on child identifies them in the old way, the other in the new way:

Child One: "Acute, acute ..... right, obtuse .... straight-silence ..whole
Child Two: "Convex, convex ...convex, convex .. silence,reflex ..silence

Note: With this exercise the child realizes that the straight angle and the whole angle do not fit the definition of convex or reflex angles.

F. Research in the environment

Here always the classification will depend on one's point of view. which is more common in the environment; make a list
convex: the corner of a table, shelf or cupboard
reflex: a chair, sofa, a corner of the library or of the room
Try to form different angles using your body.

Age: After nine years

Direct Aim: Knowledge of convex and reflex angles

Indirect Aim: Preparation for convex and concave polygons

G. A new definition of an angle

Presentation: (A new definition of an angle) Invite child to draw an angle on a piece of paper, the sides reaching the edge, effectively dividing the paper in half. Review definition of a plane and on a slip of paper write, "An angle is", and below it a label reading, "a part of a plane". Planes go on for infinity but this plane is limited by what? The angle. And what constructs this angle? Two rays. Write on a new slip "limited by two rays" and place below previous slips. Can these two rays exist anywhere on the plane? No, they must share a common vertex. Write this on a final slip thus finishing the new definition: "An angle is a part of a plane, limited by two rays sharing a common point of origin". Stack sheets of paper above and below to show the child that the angle exists in that plane only.

Plane Figures

Materials: Box of sticks, supplies, board Red cord Paper and scissors, red pen Labels for writing

Presentation: The teacher places the red cord on the board so that a simple open curve is formed. What is this? It is a curve that has a beginning and an end. Which is the beginning and which is the end? I can't tell, but what is important is that my hand can go all around, inside and out without ever leaving the plane. This is an open curve; it is like an open gate.

The teacher ties the two ends of the cord and places it on the board again so that it forms a simple closed curve. Again the teacher moves her hand around the figure on the plane: outside the figure. The teacher lifts her hand and touches the plane inside the figure and identifies it - inside the region of the figure. Identify the internal and external parts of the plane. This is a closed curve region. We can also add the quality "simple" since it is not overlapping (demonstrate this) Thus, it is a simple closed curve. Set it aside.

The teacher places a single stick on the board, moves her hand all around it on the plane. A second stick is joined to this one and they are placed on the board forming any angle. The teacher's hand still can move all around the angle. A third stick is united to these two and placed on the board; the teacher moves her hand all around it. This is a broken line. It is open; it does not form a region. Since it is straight in places, it is not a curve. It is like a line that has been broken.

The three sticks are united to form a triangle. This time my hand must go around the figure, but I must lift it to go inside. Identify the internal and external regions. We can call this figure a polygon, which is just a general name.

Place the simple closed curve region (cord) and the polygon (stick) side by side on the board. The polygon is the opposite of the closed curve region. These are the two big families with which we will be working.

To reinforce the concepts just given, the teacher draws figures in red on the paper and invites the child to cut them out. (remove the cord and sticks)

We can classify all the figures into two groups.(place out a label which says figures (or regions)) Beneath this heading place the two sub-headings - closed curve regions; and polygons. Sort the four cut-outs into these two groups.

All of these are figures (regions). They can be closed curve regions, limited by a curved line. They can be polygons limited by broken lines.

Classification Exercises

Materials: Geometry cabinet insets
Additional insets for the geometry cabinet
Two concave figures constructed previously with the child now in red on laminated cardboard
Seven classification cards
A stick from the box of sticks

A. Classification of closed curve regions and polygons
The teacher places out the label "Figures or Regions" and below it: "Closed Curve Regions" and "Polygons". The child is invited to classify all of the figures into these two columns, emptying the

cabinet, drawer by drawer, placing the figures into the two groups. On a large sheet of paper the child draws the result of this classification, drawing each figure.

B. A sensorial classification of convex and concave (re-entrant)
The teacher takes any convex polygon and the stick and passes the stick over the surface of the inset saying - internal... internal... internal. The internal region is not interrupted by any part of the external region. This is a convex figure. the teacher repeats the experience with any convex closed curve figure and arrives at the same conclusions. Taking a concave polygon, the experience is repeated - internal...internal....internal... but, look... the stick touches an internal part, then an external part and then an internal part again. This is a concave (re-entrant) figure. The experience is repeated with a concave closed curve region, and the same conclusions are made. The teacher places out the label "Figures (Regions)" and below it "convex" and "concave (Re-Entrant)". The child is invited to classify these four figures. Notice that in each column there is one closed curve figure and one polygon. the child continues classifying all of the other figures. All of the figures will end up in the convex group except the three flowers. These three figures are a special case. It seems that we should classify these flowers as concave, but concave flowers have a very different shape (this will be shown later). these flowers are convex. The child copies the situation on a large piece of paper.

Note: At a later age, these figures will be classified according to the presence of convex or reflex angles.

C. Closed curve regions/polygons - convex/concave
The teacher places out the heading card - "Figures (Region)" and below it - "Closed Curve Regions" and "Polygons". Below each of these subheadings is placed "Convex" and "Concave (Re-Entrant)". the child classifies each figure, making four columns. The child copies the situation on a large piece of paper.

At the end the teacher isolates the concave figures and their classification cards. We will only be interested in convex figures. When we talk about closed curve regions and polygons, we can assume from now on that they will be convex. One exception will be considered later.

Polygons - Triangles

A. Triangles - Classification

Materials: Box of sticks, supplies, board
Measuring angle (additional materials listed later)

Presentation: The teacher chooses three sticks randomly - a, b, c. The second group of sticks containing a, a, b. A third group of sticks are all equal - a, a, a. Invite the child to unite the sticks of each group to form a polygon. Notice the color of the sticks. These triangles are all related. Identify each triangle. This is a scalene triangle, because all the sides are different. This is an isosceles triangle, because two sides are the same. This is an equilateral triangle, because all three sides are the same.

Note: The child already knows the names and the etymology. The new experience here is the construction of the triangles.

The isosceles triangle has two equal sides. The equilateral triangle has two plus one... three equal sides. This means that the equilateral triangle is also isosceles, but an isosceles triangle is not equilateral. An equilateral triangle has two equal sides like the isosceles triangle, but it has something more - the third side is also equal, thus it is equilateral. Set the three triangles aside.
The teacher takes two sticks: orange (6cm) and red (8cm). These are united and placed on the board to form a right angle. Use the measuring angle to verify this, and leave the measuring angle in position. Let's find the stick that will join these two sticks, without altering the position of the first two sticks, without altering the angle. The stick that fits is the black (10cm). Join the three sticks to

form a triangle and place it on top of the measuring angle.

Note: This 6, 8, 10 triangle is one of the Pythagorean triples - a series of three numbers which satisfies the Pythagorean theorem: a2 + b2 = c2. The Egyptians discovered the first triple: 3, 4, 5. With the box of sticks only 6, 8, 10 and 12, 16, 20 are possible (others: 5, 12, 13 and 8, 15, 17).

The teacher takes two different sticks and unites them to form an angle greater than the measuring angle. The third stick is found to unite them. After checking the angle to be sure that it is greater than the measuring angle, remove the measuring angle.

The teacher takes three different sticks and unites them. With the measuring angle and acute angle is formed, and the third side is added and united. This time we must check the other two angles to be sure that they are also acute angles. Measuring just one angle is not enough. Since all three of the angles are smaller than the measuring angle, that is, acute angles, this is an acute-angled triangle.

Identify the others as well, stating the number of characteristic angles. Note that the right-angled triangle and the obtuse angled triangle each had two acute angles. An acute-angled triangle however must have three acute angles.

Materials:
Geometry cabinet drawer of triangles, reading labels
Paper for making labels
Second level reading labels:
right-angled scalene triangle
obtuse-angled scalene triangle
acute-angled scalene triangle
right-angled isosceles triangle
obtuse-angled isosceles triangle
acute-angled isosceles triangle

The triangles just constructed are lain out in two rows as they are in the drawer of triangles: the top row is classified according to sides, and the second row according to angles.

Every triangle by its nature has two qualities. One quality refers to its sides and the other refers to its angles. If I ask you to draw a right-angled triangle; the triangle you construct will also be isosceles or scalene. Let's label these triangles as we have always known them (using the geometry cabinet labels).

For each of these triangles we must add another quality. The first triangles has been classified according to its sides (copy the name onto a paper label). Let's classify it according to its angles. Use the measuring angle to determine the classification. Write a new label. Place the two labels on top of the figure. Do the same for the other two figures. Use a protractor to measure the angles to determine that all of them are equal. Therefore it is also equiangular. Continue with the triangles that were classified according to their angles. Classify them according to their sides determining this by the sticks which are different colors; thus the sides are different lengths. Write two labels for each, as before. Now we no longer have some triangles classified according to sides and others according to angles. All have been classified by both characteristics.

Look over the triangles to see if there are any duplicates. remove one acute-angled scalene triangle and its labels. There are no other duplicates.

There is something missing from our series. The teacher constructs an obtuse-angled isosceles triangle. The child is invited to classify it according to its sides (write a label) and then by its sides (write another label). Be sure it is not a duplicate. There is only one other triangle missing. Invite the child to unite two like sticks and to form a right angle with the measuring angle. Leave the measuring angle in its position as you try to find a stick which will unite them. Allow the child to try a second pair. It is impossible.

Introduce the neutral sticks. Invite the child to lay them in a stair. This stick (indicating the first) will be used to close a right angle formed by a pair of sticks of the first series. Place a colored stick next to it. Go on up to ten.

Invite the child to try the neutral stick that corresponds to his pair. Unite the sticks and classify the triangle. Be sure it is not a duplicate.
There are no other triangles in reality. Arrange the triangles in three columns - scalene, isosceles, and equilateral; then in three rows - right-angled, obtuse-angled, and acute-angled.

Isolate the first triangle and place its labels below it as you read them. Ask the child to identify the functions of each word; there are two different adjectives and two like nouns. Since we only have one triangle we can eliminate one of these nouns. Tear off the word "triangle" from "right-angled triangle". Place both adjectives in front of the noun and read the whole thing. Copy this onto one new long label. Remove the old labels. Continue in the same way for the others. The adjective describing the sides is always closest to the noun because it is the most important.

With the equilateral triangle, isolate the three adjectives and discard two of the nouns. Begin with "acute" and see if that quality would precisely indicate this triangle. No, there are other acute-angled triangles. With equiangular and equilateral, each one by itself is sufficient to identify this triangle. We'll use the more common one. We can remember that equiangular refers to the same triangle.
Take the insets from the drawer of triangles and match them one by one to the figures made with sticks. Identify each inset as it was previously known and give it its second quality. By passing the inset through its frame backwards, we can prove that it is isosceles.
Two triangles of the geometry cabinet correspond to the same triangle, so we can eliminate one inset and its frame. However, there are two triangles which have no inset to match. Bring out the additional insets. Invite the child to identify each using the measuring angle. Use the surface cards as you would use the frame to classify the sides.

These are all of the triangles of the 6 - 9 classroom. We also have labels for the 6 - 9. Invite the child to match new labels as he puts the triangle insets away. "Equilateral" is an old label, because the name didn't change.

Exercise: Trace each inset, copy its name and write the reason for its name.

B. Triangles - Parts of the triangle

Materials: Seven triangles constructed previously Drawer of triangles, now including two additional inset All other triangles in the environment Box of sticks, supplies, including perpendicular angle Triangle stand

Presentation: The teacher takes the equilateral triangle as a first example. Touching the surface, the teacher says - surface. The layer paint gives the concept of the surface.

Using the corresponding triangle of sticks, the teacher runs her finger around the perimeter. The sticks are the image of the perimeter (perimeter: Greek *peri*, about, and *metron*, measure; thus the distance around). the teacher indicates each side, naming each "side". At the end of the plural form is given "sides". The angles and vertices are identified in the same way. The triangle inset is standing on its side, perpendicular to the table as the teacher identifies "base:; the triangle is turned to identify each new base. The plural form is given "bases".

Note: This is to prepare for the conclusion - that any side can function as a base.

Place the triangle inset upright in the stand with reverse side closest to the groove and facing the children. Hang the plumb line in the groove to that only the cord is visible. Move the plumb line along until the cord meets the vertex opposite the base. Holding the line at the vertex so that a line segment is formed, the teacher says "height". This is the height of the triangle in relation to this base. Indicate the base in the groove.

Take out the perpendicular angle and identify it. Place it in the groove at the center of the triangle so that one side will coincide with the plumb line. Repeat the identification of the height. since the height coincides, we can say that the height is perpendicular to the base.

How many bases are there? Three. How many heights? Three, same as the bases. How many sides? Three. We can conclude that the number of heights is equal to the number of bases which are the same as the sides. The common characteristics of the height is that each is perpendicular to its relative base. Continue identifying the parts: perpendicular bisector, median (medians), angle bisector (angle bisectors).

Explore the nomenclature of other triangles. For two of the three bases of the obtuse-angled triangle, the height is external and is perpendicular to the extension of the base. For two of the three bases of the right-angled triangle, the height coincides with a side.

Note: This is why the initial presentation should deal with an acute-angled triangle.

C. Special nomenclature of the right-angled triangle

Presentation: Take the two right-angled triangles and identify their parts as before. In identifying its sides, we can give these sides particular names.

Right-angled scalene triangle - The side which is opposite the right-angle is called the hypotenuse. The other two bear the Greek name cathetus. One is longer (major cathetus); the other is the shorter (minor cathetus). All of the other nomenclature is the same.

Right-angled isosceles triangle - The longest side which is opposite the right angle is called the hypotenuse. the other two sides bear the name cathetus. since they are equal, there is no distinction between them.

Note: The presentation which should follow is "Points of Concurrency" which would be a study of the meeting point of the the three heights, that of the three medians, and the orthocenters.

Study of Quadrilaterals

A. Classification

Materials: Box of sticks, supplies

Presentation: The teacher constructs the first figure using a yellow (20cm), a brown (12cm), a pink (8), and an orange (6 cm), uniting them so that the yellow and orange are not consecutive sides and are not parallel. This is a common quadrilateral (trapezium). the child identifies its principal characteristic; it has four sides (quadrilateral: Latin quadras < quator, side, and lateris, side).

Invite the child to build a figure exactly like this one, of sticks. Superimpose the second one on the first to show that they are equal. Move the two sides back and forth like the arms of a balance, stopping where the yellow and orange sticks are parallel. Use two small sticks as guides to check if they are parallel. This is a trapezoid. It is a quadrilateral, but it has something more. It is a quadrilateral which has at least one pair of parallel sides.

Note: It is important for the child to understand this concept; a trapezoid is a quadrilateral, but a quadrilateral is not necessarily a trapezoid.

Invite the child to choose two pairs of like sticks; join them to form two angles, then unite them to form a quadrilateral in such a way that two sticks of the same color do not touch. This is a common parallelogram. It has two pairs of parallel sides (indicate these pairs). The common parallelogram is a quadrilateral with two pairs of parallel sides. Is it also a trapezoid? Yes, but it has something more. Invite the child to construct with the sticks the same figure as before - the parallelogram. Superimpose this figure on the other to show that they are equal. Stand the figure on end and using the measuring angle as a guide, straighten the sides until the side coincides with the measuring angle. If one angle of this quadrilateral is a right angle, then all of the others will also be right angles. This is a rectangle. It is a parallelogram with all right angles. The previous figure was just a common parallelogram, whereas this parallelogram is no longer common. It has all right angles.

Is the rectangle a parallelogram? Yes, it has two pairs of parallel sides. Is the rectangle a trapezoid? Yes, it has at least one pair of parallel sides. Is it a quadrilateral? Yes. Is the common parallelogram a rectangle? No.

Invite the child to unite four like sticks. This is a rhombus. It is a parallelogram with equal sides. Is the rhombus also a rectangle? No. Is it a parallelogram? Yes. Is it a common parallelogram? No. Is it a trapezoid? Yes. Is it a quadrilateral? Yes. Is the rectangle a rhombus? No.

Invite the child to reproduce the previous figure. Superimpose one on the other to show that they are equal. As before, stand the figure on end to straighten its sides, using the measuring angle as a gauge. This is a square. It is a parallelogram ... (is that all I need to say? No) .... with equal sides ... (is that enough?) .... with right angles. Repeat the definition. By saying that it has equal sides, the common parallelogram and rectangle are excluded. In order to exclude the possibility of being a rhombus, however, another quality must be added - right angles. Questions similar to those for the rhombus may be posed.

These are all the quadrilaterals of reality.

B. Quadrilaterals - Types of trapezoids

Materials: Box of sticks, supplies Geometry cabinet insets, reading labels -"square", "rhombus", "rectangle", "trapezoid" Additional insets: common quadrilateral, common parallelogram, three trapezoids (all except the equilateral) Second level reading labels - "common quadrilateral", "common parallelogram", "isosceles trapezoid", "scalene trapezoid", "right-angled trapezoid", "scalene trapezoid", "right-angled trapezoid", "obtuse-angled trapezoid"

Presentation: Isolate the trapezoid previously constructed with sticks. In order to extend our family of trapezoids, we must first know its parts. The two sides which are parallel are bases. Identify the other two as sides. Are the sides equal? No. This is a scalene trapezoid. Recall the meaning of scalene from the triangles.

Invite the child to take four sticks, two of which are equal. Unite them so that the two sticks are not touching each other. Arrange them so that the two bases are parallel. Identify the two bases and the two sides. Are the two sides equal? Yes. This is an isosceles trapezoid.

The teacher takes four sticks (yellow - 20cm, black - 10cm, orange - 6cm, brown 12cm) and unites them so that the yellow and black sticks form the parallel sides. Identify the bases and the sides. Use the measuring angle to show that a right angle is formed by the perpendicular side. This is a right-angled trapezoid.

The teacher takes four sticks (yellow - 20cm, green - 14cm, red - 8cm, brown 12cm) and unites them so that the yellow and green sticks form the parallel sides. Use the measuring angle to identify the two obtuse angles. They are opposite each other. This is an obtuse-angled trapezoid.

Note: It is not so much the number of obtuse angles, but the opposite position of the obtuse angles which is important. This is the only trapezoid which can be divided into two obtuse-angled triangles.

Bring out all of the quadrilateral formed with sticks (common quadrilateral, the trapezoids, common parallelogram, rectangle, rhombus, square) and organize them as the child recalls the name of each. Bring out the old labels and match them to the figures, This old label "trapezoid" will be of no use anymore, and discard it. Bring out the new labels and match them accordingly. Each member of the family of trapezoids has its own characteristic; therefore they each have a different name.

Match the insets of the geometry cabinet with the figures: square, rectangle, rhombus. Before the trapezoid was known only as a trapezoid. Identify its other characteristic: isosceles. Bring out the additional insets and match them as well.

C. Quadrilaterals - Classification according to set theory

Materials: All the quadrilateral insets (of previous presentation)
Board covered with paper
Blank labels, pen

Presentation: Like the portrait gallery to be found in the houses of noble families, the family of quadrilaterals has its own gallery, which has six portraits of the six members of the quadrilateral family. But instead of walking with our feet, we have strolled with our minds through this gallery of the quadrilateral. Now that we have examined these figures one by one, face by face, we must look at this family of quadrilaterals together, as a whole. We must look at the interrelationship of the members of this family. As we stroll along, we notice similar features in their faces. But in the family of quadrilaterals, we see increasing perfection as we go through the generations. This is like looking at the genealogical tree of the family. With the last descendant of this family the square, we have the perfect quadrilateral. It is the only quadrilateral which is a regular polygon. In this family of geometry we see increasing perfection in the last descendants. This is in contrast to some noble families where we see degeneration in the last descendants.

Place all of the insets on the board in a random group. Draw a circle around the figures. This is a gallery. All of these portraits belong to the family of quadrilaterals. Write a label "quadrilateral" and place it inside the circle. They are all quadrilaterals because they all have four sides.

Isolate the common quadrilateral and the label to one side of the circle, draw a circle around the remaining figures. These are all trapezoids. Place a label inside the circle. "trapezoid" All of these quadrilaterals have at least one pair of parallel sides.

Isolate the four trapezoids and the label to one side of the circle and draw a line around the remaining figures. These are all parallelograms. Place a label in the circle. All have two pairs of parallel sides.

Isolate the square and the rectangle and draw a circle around them. These are rectangles. Place a label in the circle. They are parallelograms with four right angles. Recall that a square was included in the drawer of rectangles.

Place the rhombus and draw a circle in such a way that the rhombus and the square are included in the circle. These are rhombi. Place the label on the rhombus side of the circle. They are parallelograms with four equal sides.

Place a label for the square in the intersection of these two sets. This means that the intersection of the set of rhombi and rectangles is the square.

It is interesting that at the end of our visit to this portrait gallery, we see one family member who is perfect and marvellous. His portrait is placed between the portraits of his father and his mother. As it sometimes happens in nature, this child has inherited the best qualities of both parents.

In the set of quadrilaterals, the child is the square. His father and mother are the rectangle and the rhombus. The square has the distinctive characteristics of the rhombus: equal sides. The square is the perfect quadrilateral; it is the only regular polygon among all of the quadrilaterals.

D. Parts of the Quadrilateral

Notes: The nomenclature of quadrilaterals, and other polygons with more than four sides, will present some difficulties, because, unlike the triangles, not all quadrilaterals have the same nomenclature. Also the stick figures must be used in order to identify the diagonal.

Materials: Quadrilaterals made previously with sticks
Insets of the quadrilaterals
Stand, small plumb line, perpendicular angle
Box of sticks, supplies

Presentation: (Square). Isolate the square made of sticks and invite the child to name it and recall its characteristics. Let's examine the parts of this figure. the teacher points to each part as it is named: Surface ...perimeter... Where there are more than one the teacher names and identifies each singularly, then gives the plural ... side (sides), angle (angles), vertex, (vertices), base (bases).. to identify the base, stand the figure on its side. We can conclude that each side can serve as a base. Repeat with the inset.

Place the figure in the stand and drop the plumb line into the opening. Move the plumb line along until it coincides with one of the sides ... "height" ... Continue moving the plumb line along whispering ... "height, height, height" ..... when the plumb line coincides with the other side, say aloud, "height". Place the perpendicular angle beside the figure to determine that the height is perpendicular to the base. The principle heights are those two on the sides. all the segments in between are also heights but in reality they are all the same height; they are all equal. Repeat the experience with other bases.

Did you notice that I had to concentrate on keeping the square erect in this stand? If I let it slant, it is no longer a square. This figure needs a support. Line up the neutral sticks and find the one that corresponds to the sticks used for this square. Join it at two opposite vertices. This segment which connects two non-consecutive vertices is the diagonal (diagonal: Greek *dia*, through, and *gonia*, angle; thus the line that goes through the angle). We can say that for the quadrilaterals the diagonal is one of the most important elements.

Note: Dr. Montessori suggests that at this point we introduce the story of construction to show the importance of the diagonal. It keeps the roof from opening up.

Notice that the diagonal divides the square into two triangles. The triangle is the constructor of this figure. Place another stick along the other diagonal to show that the square is divided into four triangles. Again the triangle is the constructor.

(Rhombus). As before use the stick figure to identify ... surface, perimeter, side (sides), angle (angles), vertex (vertices), base (bases) ... In identifying the height, proceed as before this time staying silent until the plumb line coincides with the first top vertex ... "height". Continue, whispering "height, height, height", until the plumb line coincides with the second bottom vertex. silence prevails until the plumb line coincides with the extreme top vertex ... "height". All of these heights are equal. The first principle one is internal. The second was external, and was perpendicular to the extension of the base. These are relative to this base.

Repeat the experience with the other bases and those of the inset. Lastly, identify the diagonals.

(Rectangle). Identify: surface, perimeter, side (sides), angle (angles), vertex (vertices), base (bases) ... These four bases are equal in pairs. Identify the height as for the square. Notice that the heights are equal in pairs. If the base is long, the height is short, and if the base is short, the height is long. How many heights are there? Two. Identify the diagonals.

(Common Parallelogram). Identify the same parts as before. Notice that the bases are equal in pairs. Identify the heights as for the rhombus, and make the relative observations as for the rectangle. How many different heights are there? Two. Identify the diagonals.
(Trapezoid). Begin with the most general: the scalene trapezoid. Examine the parts as before: surface, perimeter, side (sides), angle (angles), vertex (vertices), diagonal (diagonals). Identify the two sides which can be bases. They are parallel. The other two can never be bases. Notice that in all of the parallelograms all of the sides served as bases. here, however, only two of the four sides can serve as bases.

Identify the longer (larger) base and the shorter (smaller) base. The other two sides which are not bases are called legs. (Recall the etymology and the story of the Greek table)

Since the number of bases has decreased in relation to the number of sides, the number of heights will also decreases. Identify the height as before, saying "height" when the plumb line coincides with the first top vertex, whispering "height, height" as it moves along, saying "height" as it coincides with the second top vertex and silence as it continues along. Repeat with the other base. How many heights are there? Only one, for they are all equal.

Identify median
- the line segment which connects the midpoints of the two legs; and joining line of the midpoints of the parallel sides
- the line segment which connects the midpoints of the bases.

Identify the nomenclature of the isosceles trapezoid in the same way. Notice that the legs are equal. The joining line of the midpoints of the parallel sides corresponds to one of the heights.

Identify the nomenclature of the right-angled trapezoid. One of the legs is perpendicular, thus it corresponds to the height.
The nomenclature of the obtuse-angled trapezoid is the same as before. Since one height is external, identify the heights for the rhombus.

Note: For the following presentation, the common quadrilateral was enlarged so that the longest side was 20cm. #1 - plain, #2 - one diagonal drawn on each side, #3 - cut along one diagonal resulting in two triangles with the black line along one side each.

(Common Quadrilateral). Identify each part as before: surface, perimeter, side (sides), angle (angles), vertex (vertices), diagonal (diagonals) (use #2). seeing that nothing is new, the teacher asks after identifying each part - "Where's the difficulty"?

Stand up the figure. Where is the base? Is it this? Try all four of them. We'll realize which is the base when we find the height. Discover that each base has two different heights. It can't be! The secret of this figure is that it has no base and therefore it has no height.

(Solving for the area). When we calculate the area we must multiply the base times the height. How can we calculate the area of this figure which has no base and no height? This is where the triangle becomes important. We must decompose the quadrilateral into triangles by tracing the diagonal (use #2, then #3, superimpose the triangles on the quadrilateral to show equivalence). With these triangles, we have no problem calculating the area because each triangle has as many heights as bases. Calculate the area of each and add.

We can conclude that the common quadrilateral has neither a base, nor a height, thus the diagonal has an important role. When we must calculate the area of such a figure, we must divide the figure along the diagonal.

Polygons with More Than 4 Sides

A. Parts of Polygons

Materials: Box of sticks, supplies "Polygons" drawer of the geometry cabinet

Presentation: Invite the child to choose any five sticks and write them. The teacher arranges the figures so that it is convex. Ask the child to count the sides and identify the figure - 5, pentagon. Give the nomenclature as before: surface, perimeter, side (sides),angle (angles), vertex (vertices). In order to make this figure stable the teacher attaches one stick and identifies it as a diagonal. Since it still is not stable, the teacher adds a second diagonal.

Notice that a triangle did not met a diagonal. The quadrilateral, having four sides (one more than the triangle) needed one diagonal to make it stable. This divided the quadrilateral into two constructive triangles. Here with a pentagon (having one more side than the quadrilateral) two diagonals were needed. They divided the figure into 3 constructive triangles. For a hexagon 3 diagonals will be needed.

Identify one angle. Next to it is a successive angle. The next angle is not successive, and there is a diagonal. The next angle is also not successive, and there is the other diagonal. The last angle is a succeeding angle. A line connecting this (first) angle with a succeeding angle is merely a side. However, a line connecting this angle with a non-successive angle forms a diagonal.
Examine the other angles as a focal point. Each time it is possible to make two diagonals, and three constructive triangles.
Take the polygon from its drawer. Ask the child to identify it and its particular characteristic - pentagon, having equal sides (turn it in its frame as proof) thus, a regular pentagon.

Repeat the nomenclature as before; everything is the same. There are three other parts which only pertain to the regular polygon. This knob indicates the center of the figure. A line form the center to any vertex is a radius. Even a line from the center to the midpoint of any side is also a radius, but it has a special name: apothem. Three period lesson.

Examine the other figures as well.

Exercises:
classified nomenclature
command cards

Comparative Examination of Polygons

(Triangle). What is the distinctive characteristic of the triangle? what did it have? Everything. This is because the triangle is the constructor. The only thing the triangle did not have was a diagonal. It does not need a diagonal because it is stable without it. All of the angles are successive.

(Quadrilaterals). The common most fundamental characteristic of quadrilaterals is their diagonal. All, except the common quadrilateral, had bases, and therefore they had heights. the common quadrilateral has no base, thus no height. It had to be considered in terms of the triangles formed by the diagonal.

(Polygons). The polygons have no base and therefore, no height. The regular polygon had improved nomenclature - center, radius, apothem. this is true also for the equilateral triangle and the square, because they are also regular polygons.

A. Study of the apothem

Intuition of the apothem
This was given when the child studied the Parts of the Polygons. The regular polygons had three elements of nomenclature that was characteristic: the center, the radius, and a special radius called the apothem. The presence of two different radii indicates that two circles are also involved.

Identification of the apothem

Materials:
Geometry cabinet: presentation tray, drawer of polygons Inset of a triangle inscribed in a circle
Fraction inset of the square divided into four triangles
A special cardboard square (10cm diagonal and black dot in center
Box, entitled "Apothem", containing white cardboard circles, each radius is drawn in red and each circle corresponds to a regular polygon:

Polygon Sides Radius

triangle 3 approx. 2.5cm
square 4 approx. 3.5
pentagon 5 approx. 4.0
hexagon 6 approx. 4.3
heptagon 7 approx. 4.5
octagon 8 approx. 4.6
nonagon 9 approx. 4.7
decagon 10 approx. 4.8

Note: All of the measurements of the radii are irrational numbers with the exception of that which is relative to the equilateral triangle. In that case, the radius of the inscribed circle is half of the radius of the circumscribing circle. Theorem: The area of the circumscribing circle is four times the area of the inscribed circle.

Note: The two new figures of the equilateral triangle and the square are introduced, because those same figures found in the presentation tray have sides of 10cm. Therefore it is impossible for the two original figures to be circumscribed by a circle which has a diameter of 10 cm.

Presentation: Recall the special nomenclature pertinent to regular polygons: center, radius, and a special radius called the apothem. Now we'll find out what's special about the apothem. The child may remember these three elements of nomenclature, but at this point he hasn't understood the reasoning.

Present the inset of the equilateral triangle inscribed in a circle. remove the circle insets and invite the child to classify the triangle. Verify that the sides are equal by tracing one side on a piece of paper and matching the other sides. Isolate the triangle at center stage left and get rid of the frame and circle segments.

Bring out the square fraction inset. Lift our two opposite triangle pieces and juxtapose to form a square. Take the red cardboard square. Superimpose the fraction pieces to demonstrate congruency. Place the cardboard square next to the triangle and get rid of the inset, since these constructive triangles have served their purpose.

Bring out the drawer of polygons and ask the child to line up the polygons (in a row with the triangle and square) naming them as he goes along.

We have eight regular polygons, all of which have a center, a radius, and a special radius - the apothem. Since there are two different radii, there must be two different circles. Bring out the circle inset and frame from the presentation tray, and place it above the series of polygons. This is the first of the two circles. this series contains the second circle. Invite the child to put the circles in order from smallest to largest. Place each circle under a polygon. How interesting! The first circle is common to all polygons, but each polygon has its own circle. The triangle has this circle (indicate the circle inset) and this smaller circle (indicate the corresponding small cardboard circle) the square has ... and so on, for all polygons.

B. The polygon in relation to the first circle

Remove the inset of the circle from its frame. Place the inset of the triangle in this frame. The circle holds it inside (circumscribe: Latin *circum* - around, *scribere* - write or draw). Place the two insets back to back to see this in another way. Hold the two insets back to back using two fingers of one hand on the little knobs. We can see that the center of the triangle coincides with the center of the circumscribing circle.

Place the triangle inset in the circle frame. Now we can try to find the radius. It is the segment which joins the center to one of the vertices. This radius is also the radius of the circumscribing circle. The teacher identifies the radius of the figure, then removes the triangle from the frame, inviting the child to identify the radius without the help of the circle.

C. The polygon in relation to the 2nd circle

Take the white cardboard circle and superimpose it on the back of the triangle inset so that it is inscribed. Then rotate the circle so that the radius is perpendicular to the side of the triangle.

Because this line segment is perpendicular, it meets the midpoint of the side. This circle is inscribed by the triangle; it is contained within the triangle. Hold the triangle and circle as before to show that the centers coincide.

This line segment which joins the center of the polygon with the midpoint of one of the sides is that special radius called the apothem. It is also the radius os the inscribed circle.

(apothem: from a Greek verb meaning to bring down, thus the line segment is brought down (dropped) from the center of a regular polygon to one of its sides)

Ask the child to identify the apothem without the aid of the circle.

D. The polygon in relation to the 1st and 2nd circles simultaneously

We've seen that the triangle is embraced by the first circle and it embraces the second circle. Place the inset in the frame wrong side up (extend off the edge of a table so the triangle is flush with the frame) and place the small circle on the triangle inset. We know that the three centers coincide; they have become one center.

Repeat the same procedure with all of the other seven polygons. After we can bring these facts to the child's attention: 1) Each time the number of sides of the regular polygon increases, so too, the size of the inscribed circle increases. 2) If the size of the circle increases, the length of the radius increases. The length of the radius varies from a minimum of 25 cm for the equilateral triangle to a maximum of just under 5 cm for the decagon. 3) As the number of sides of the regular polygon approaches infinity, as the inscribed circle becomes larger, the radius approached 5 cm the radius of the circumscribing circle. 4) When the inscribed circle of the polygon coincides with the circumscribing circle of the same polygon, the polygon no longer exists; it is identified in the circles. 5) The two radii coincide, because since there is no polygon; the circle are one. The radius of a circle can be regarded as the apothem of a polygon having an infinite number of sides,

Age: after 9 years

From Irregular to Regular Polygons

Materials: Box "regular and irregular polygons" which contains:
Six special measuring angles: 1080, 1200, 1280 (approximately), 1350, 1400, 1440
Reading labels "equilateral and equiangular polygon", non- equilateral and non-equiangular polygons, equiangular but non- equilateral polygon, equilateral but non-equiangular polygon, regular polygon, irregular polygon, non- equilateral polygon, non-equiangular polygon, equilateral polygon, equiangular polygon
Cardboard sticks (40 cm long) with a hole at one end only
Box of sticks and supplies, measuring angle
Geometry cabinet drawer of "polygons
Cards for 5-10, which have the greek roots on the other side

Note: For the child to understand the characteristics which determine regularity or irregularity of a polygon, we must examine more than one family of polygons, since the triangle will not be sufficient.

Presentation: Invite the child to build an acute-angled scalene triangle or any other triangle he'd like except the equilateral triangle. Classify the triangle which was constructed by calling attention to whether its sides or its angles are equal: Are the sides equal? no, this is a non-equilateral triangle. (write the label in black: non-equilateral triangle) Are the angles equal? no, this is a non-equiangular triangle. (write another label)

Invite the child to construct with the sticks the triangle which he was forbidden to build before. Ask the same questions and write the two separate labels: equilateral triangle, equiangular triangle. Now each triangle has two labels: one which refers to the classification of its sides and one which refers to its angles. Read them. We want to make one label for each. Tear off "triangle" on one of them. Read what's left. We need the word "and". Write "and" in red and place it between the adjectives. Repeat the experience for the second. Now we can make new labels for each one (no color distinction for and anymore). Read the labels. "non-equilateral and non-equiangular triangle", "equilateral and equiangular triangle". All of the goodness is in one, while all the badness is in the other. Two negative qualities give an irregular triangle. Two positive qualities give a regular triangle. Set these figures aside.

Invite the child to construct a common quadrilateral, any trapezoid, or a common parallelogram. As before classify the figure by asking: Are the sides equal? Are the angles equal? After each answer, make the appropriate classification and write a label. Unite the two adjectives with and in red as before. Read the parts and rewrite a new label (all in one color): non-equilateral and non-equiangular quadrilateral.

Invite the child to build a rectangle. Use the measuring angle to verify that one angle is a right angle, therefore all are right angles. Classify the figure as before writing two labels. Notice that this is the first time a figure has one positive quality and one negative quality. Tear off the adjectives and eliminate one noun. Decide on which one shall come first. Read the two adjectives; we need the work but this time. Write but in red and place it between the adjectives. Read it and write new label: equiangular but non-equilateral quadrilateral.

Note: this use of conjunctions *and*, *but* will aid the development of set theory.

Invite the child to build a rhombus. Proceed as before. The label reads equilateral but non-equiangular quadrilateral.
Lastly the child constructs a square. Proceed as before. Equilateral and equiangular quadrilateral.

Align the four quadrilaterals and their labels in the order of which they were presented. Two negative qualities produce an irregular quadrilateral. One positive quality and one negative quality still produce an irregular quadrilateral. Two positive qualities produce a regular quadrilateral. Bring to the child's attention the progression towards perfection.

Place the six figures just built - triangles and quadrilaterals on the table with labels. In both families we have the two extremes. Each family has two opposite figures: one has two negative qualities, the other has two positive qualities. But only the family of quadrilaterals has two intermediary figures which constitute the passage from imperfect to imperfect.

At this point we can begin to make some generalizations: a polygon is regular when it is equilateral and equiangular at the same time.
Game: Examine the quadrilaterals. Why is this common quadrilateral not regular? This rectangle seems to be regular for it has equal angles. (mention this positive quality first to show the move toward perfection) Have the child find the bad quality - the sides are not equal. The sides are equal in pairs. That's not enough. repeat the procedure with the rhombus. Finally the last one is perfect; it is regular.

Ask the child to choose five sticks at random and join them. Identify the figure: 5 sides, pentagon. As before classify the figure, write the two separate labels. Unite the qualities with and in red. Rewrite the label non-equilateral and non-equiangular pentagon". Invite the child to unite five equal sticks and arrange the figure so that it is not equilateral. Proceed as before. The last label will read equilateral but non-equiangular pentagon.

The third figure which must have the opposite qualities of the last, will be more difficult to build. Present the special measuring angles. Using the drawer of pentagons insets, we can find the figure to go with each measuring angle. Lay out the numeral cards 5-10 in a row. Take any measuring angle and place it on the back of an inset, matching the angles. The child copies this information into his notebook.

Remove all of the measuring angles except that which pertains to the pentagon. Ask the child to take these sticks: red(8), black(10), brown(12), pink(16), blue(18); and unite them :red-pink-brown-black-blue. Check their angles with the pentagon measuring angle. Classify this figure as before. equiangular but non-equilateral pentagon. Ask the child to unite five equal sticks. Control the angles with the appropriate measuring angle. Proceed as before. equilateral and equiangular pentagon.

Observe that in the family of pentagons, the same thing happened as with the family of quadrilaterals. There are two extremes and two mediators. Identify the irregular and regular pentagons. The two intermediary figures demonstrate the passage from imperfect to perfect.

Proceed as before. In the examination of polygons having more than four sides, we use one with an odd number of sides and one with an even number of sides.

Notes: When the child is working alone, he may not remember the colors of the sticks for the third figure, or their order. For this reason the cardboard sticks are provided.

Unite two sticks and use the measuring angle to form the desired angle. As each successive stick is added, check the angle formed with the measuring angle. For the last side, try the sticks from the box. If one cannot be found, measure off a cardboard stick. Cut it, punch a hole and attach it. Check the angles with the measuring angle.

Materials for Exercises:
additional inset figures
envelope containing cardboard figures
"convex polygons"
3 irregular pentagons, hexagons, octagons, nonagons,decagons: one for each of the irregular classifications
box "regular and irregular polygons"
several cords or circumferences (for making sets)

First Exercise: Ask the child to read the four long labels; observe that "polygon" has ben substituted for the name of the family (triangle, quadrilateral, etc.)

Isolate two of these labels "equilateral and equiangular polygon", non-equilateral and non-equiangular polygon". Take all of the triangles (from cabinet and from the box of additional insets) and classify them, making two groups under these two headings. Since two equilateral triangles are identical, one can be removed. Also one of the acute-angled scalene triangles can be eliminated. Bring out the two smaller reading labels - irregular polygons and regular polygons and place them accordingly above the headings. The child can copy this into his notebook, tracing the insets, or substituting the reading labels and making two lists.

Proceed with the family of quadrilaterals. This time all four long labels will be needed. Place them in order: negative, 2 mediators, positive. The child takes out all of the insets and classifies them. (eliminate one square) Place the two small labels above the headings. "Irregular polygons" includes the first three columns.

Proceed with the pentagons. The four labels are needed again. But there is only one inset. Classify it. Bring out the envelope of Convex Polygons, and invite the child to find more pentagons. Use the measuring angle to verify classification. Place the two small labels above the headings. Proceed with the other polygons.

Second Exercise: (Triangles) Place the two circumferences on the table side by side. Place the labels "equilateral polygon", equiangular polygon" in one circle, and "non-equilateral polygon", "non-equiangular" in the other. Invite the child to place all of the triangle insets in their respective places (again eliminating the duplicates). Only one is perfect. Superimpose the two circumferences to make an area of intersection. What triangle has the qualities of both sets? None. Why? Because there are no mediators in the triangle family. Place the labels "regular polygon", "irregular polygon" appropriately.

(Quadrilaterals) With the circumferences side by side, place the labels "equilateral polygon", equiangular polygon", one in each circle. Classify all of the quadrilaterals (excluding the duplicate rectangles). Only a few of our quadrilaterals have these characteristics. One figure is found in both sets - the square. Superimpose the circles and place one square in the intersection; eliminate the other. Place the labels "regular polygon" - in the intersection; "irregular polygon" - so that it touches both sets, though separately.
(Pentagons, etc...) Proceed as for quadrilaterals with other polygons.

The child copies his work by tracing the insets or substituting the names of the figures.

Construction of Polygons

Note: This activity pertains to all polygons.

Materials: Different colored drinking straws, scissors, yarn, Upholsterer's needle

Exercise: Following command cards, the child constructs the figures, using diagonals as needed for stability. For sides of equal length, the child should use straws of the same color.

Circle - Level One

Materials: Box of sticks, supplies, board, red pen
Fraction insets of the circle; whole, half, one other
Inset of the triangle inscribed in a circle

Two wooden circumferences (painted embroidery hoops of two different sizes, such that the sticks may serve as radii - large red hoop 20cm in diameter; small blue hoop 12cm in diameter)

A. Circle and its parts

Presentation: The teacher takes any stick and fixes one end to the board. remember that when we constructed an angle, two sticks were needed. Here, using one stick we'll construct something different. Place the red pen in the hole and draw a circle. Indicate the internal part: this is a circle. It is the part of a plane enclosed by a very special closed curve. Identify the center of the circle (tack) and the radius (radius: Latin rod, spoke of a wheel) of the circle (stick). All of the points that make up this red line are the same distance away from the center. This is why this special closed curve line is called the circumference (circumference: Latin *circumferre*, to carry around, *circum*, around, *ferre*, bear).

Ask the child to take another stick identical to the first. Fix them together at the center and arrange them so that they are opposite rays. This is the diameter (diameter: Greek *dia*, through, *metron*, measure).

Take the longest stick from the box and place it so that it touches the circumference at its two ends. This is a cord (cord: Latin *chorda*, cord, string).

Each of the two parts of the circumference divided by this cord is an arc. This small one is an arc (arc: Latin *arcus*, bow, arch); the large one is an arc.

The diameter is a special cord because it is the longest cord possible, and it is the only cord which passes through the center of the circle. The two arcs which result from the division of the circumference by the diameter are special arcs. They are called semi-circumferences (semi: Latin half). Three period lesson.

Bring out the fraction insets and line them up left to right: whole, some fraction (3/3), halves, and the inscribed triangle. Take out the whole inset and place it on the table. This is a circle. Take out the 1/3. This is a sector (sector: Latin a cutter) of the circle. A sector is what we call each of the two parts into which the circle is divided by the two radii. Any of our fractions can serve as a sector. Perhaps the 1/2 could even be considered as a special sector. Remove the 1/2 inset. This is a semi-circle. It is exactly one-half of the circle.

Remove one of the "moon" pieces of the last inset. This is called a segment (segment: Latin *secare*, to cut) of the circle. A segment is the name given to each of the two parts resulting from the subdivision of a circle by a cord. Perhaps even the semi-circle can be considered as a segment.

Redefine sector and segment a little more precisely. This is a sector. It is formed by two radii, two radii which are not prolongations of each other. Thus the semi-circle cannot be called a sector. The segment is formed by a cord, but by a cord which does not pass through the center of the circle, that is a cord which is not the diameter. Thus the semi-circle cannot be called a segment either, but it is the limit of these two figures.

The ring of a circle is the last part of the plane enclosed between two circumferences having the same center. Three period lesson. Classified nomenclature and commands.

Age: Seven and a half years

B. Relationship between a circumference and a straight line

Presentation: Place a stick (straight line) and the large wooden circle (circumference) on opposite sides of the board. Move one toward the other, but do not let them touch. The teacher says, "external, external, external..." The straight line is external to the circumference and vice versa.

repeat the experience this time sopping when the stick touches the circumference ... external, tangent (tangent: Latin *tangere*, to touch). The line and circumference are touching each other at one point.

Repeat the experience, this time placing the stick on top of the circumference ... external, tangent, secant (secant: Latin *secans* < *seca*, to cut). The line cuts the circumference at two points. Identify the two points of intersection. Three period lesson.

Note: Even if a short stick were used, the secant would intersect the circumference at two points, because the stick represents a straight line which goes on in both directions to infinity.

Exercises: Classified Nomenclature and commands

C. Relationship between two circumferences

Presentation: Place the two circumferences on the board and repeat the experiences of the previous presentation:

external - having on points in common
tangent - having one point in common
secant - having two points in common

Return the circumference to the tangent position. Ask the child: is the one outside or inside the other? Flip one circumference over to show that they would look like otherwise. So we can say that this circumference is external; we can also say that it is tangent. These two adjectives (external, tangent) refer to the same circumference. When we combine the qualities, one adjective becomes an adverb (externally, tangent).

Flip one circumference over. They are still tangent, but now one is internal. Repeat the transition of the adjectives. These circumferences are internally tangent.

Return to the first position and repeat the experience using these new names ... external, externally tangent, internally tangent. Move the inner circumference so that it is neither tangent nor concentric. This circumference is internal. It is inside the other, but there are no points in common.

Move the inner circle so that the circumferences are concentric. Use two small sticks to check that one is equidistant from the other all the way around. This is a particular type of internal, called concentric (concentric: Latin *con*, with, together; center). They have the same center. Concentric circles are a subset of internal. Do a three-period lesson.

Exercises: Classified nomenclature and commands

Age: Eight years

D. Relationship between a circumference and a straight line

Materials:
...Box of sticks, board, measuring angle
...Circumference

Presentation: Place the circumference and a long stick to serve as the straight line on the board. Allow the child to experiment to find the stick which may serve as a radius for this circle. The child chooses the one that looks right.

Move the straight line towards the circumference, external, external ... We will use the radius to put this straight line in relation to the circumference. We'll choose a radius which is perpendicular - use the measuring angle to check. Place one finger on the straight line where a perpendicular would intersect, and place another finger on the center. When the straight line is external, the distance between the center and the straight line is greater than the radius.

Move the straight line to the tangent position. Recall the first level definition: they touch at one point. When the straight line is tangent, the distance between the center and the straight line is equal to the radius.

Repeat the experience ... external, tangent, ... secant. Recall the first level definition. When the straight line is secant, the distance between the center and the straight line is less (shorter) than the radius.

Later we'll learn how to write the symbols for these positions.

Age: Nine years

E. Relationship between two circumferences

Materials:
...Box of sticks, supplies and board
...Two circumferences
...Two charts - **Internal** and **Secant**

Presentation: Place the two circumferences on the board and ask the child to find the two sticks which will serve as radii; place these in them appropriately. Use a long stick to align the radii in the same straight line. Remove the long stick.

(External) Place a finger on each center, to show the distance between the two centers. Is this distance equal to, greater than, or less than the sum of the two radii? Place two sticks like the radii, on the board end to end to allow the child to visualize the sum of the radii. The distance is greater; by how much? find an appropriate stick.

Two circumferences are external to each other when the distance between the two centers is greater than the sum of the radii.

This can be expressed in symbols. The symbol /d/ will represent the distance between the two centers (since d is used for diameter, we'll use a small delta). R will represent the longer radius and r will represent the shorter radius.

Thus > (R + r), or we can consider the sum of the radii. Is it equal to, greater than, or less than the distance between the two centers? Less, so (R + r) < . The two statements say the same thing, but different symbols reflect the different order of the terms.

(Internal) Refer to the chart. Placing a finger on each center, the distance between the centers is shown by the green line segment. The radii are shown side by side on the chart, but the sticks may be superimposed to leave a portion of the longer stick visible as the difference. Look at the chart. Is the distance between the centers equal to, greater than, or less than the difference between the radii? Formulate a statement about internal circumference and write the symbolized form. < (R - r) and (R - r) > .

(Externally tangent) Ask the child to place the two circumferences together again using the long stick to align the radii. Place the two extra sticks end to end to show the sum of the radii. Is the distance between the two centers equal to, greater than, or less than the sum of the radii? Formulate a statement and write the symbols. = (R + r) and (R +r) = .

(Internally tangent) Invite the child to position the circumference accordingly; the radii should be superimposed. Place the two extra sticks to demonstrate the difference between the radii. Ask the question as usual, formulate a statement and write the symbols. = (R - r) and (R - r) = .

(Secant) Refer to the chart. This time we need two series of extra sticks. Place out a pair to represent the sum and a second pair to represent the difference (Notice in all other cases we used one or the other). Begin with the sum. Ask the question, formulate a statement and write it in symbols in the two ways. < (R + r)

Discard the sticks for the sum. (R + r) >

Consider the difference the same way. > (R - r)

Using these four statements we can combine them (R - r) < to make one: (R - r) < < (R + r) or (R + r) > > (R - r) .

(Concentric) Fix the two radii to the board with an upholstery tack and position the circumferences concentrically. Invite the child to use two fingers to show the distance between the two radii. the two fingers are at the same place; there is no distance. Recall the meaning of concentric. = 0, which means there is no distance. Three period lesson. Classified nomenclature and command cards.

Age: Nine years

Chapter Three - Congruency, Similarity and Equivalence

Introduction to the Material - Squares

Materials:
Insets of the fractional parts of the square
Geometry charts

Presentation: Bring out the insets arranging the frames in two rows with the whole inset centered at the left between the two rows.
Isolate the frame and inset of the whole. The child identifies it: square, whole. Continue bringing forth inset one at a time, identifying each: the number of equal parts, how the whole was divided each time. After identification of the halves, present charts 1 and 5. After the fourths have been identified, present charts 2 and 6. The eighths (rectangles) were formed by the division of each small square (1/4) by joining the midpoints of two opposite sides. The eight small triangles were formed by the two diagonals and the lines joining the midpoints of the opposite sides.

Note: If the two insets of the fourths were transparent, and were superimposed, the resulting lines would be those of the eight triangles.

Remove the 2/8 rectangles and 4/16 squares to show that the sixteenths were formed by joining the midpoints of the other opposite sides. This is a repetition of the passage from halves to to fourths. Likewise, use 2/8 triangles and 4/16 triangles to show that the sixteenths were formed by the other diagonal. Again this is a repetition of the passage from halves to fourths.

Note: For this presentation it is important that the pieces are arranged in the inset frame, just as they are shown on charts 1 - 4. Any arrangement showing an inscribed square is incorrect at this stage.

Identify the value of the pieces of each inset: whole, halves, halves, fourths, fourths, etc...
Demonstrate that the pieces are equally divided by superimposing them back to back. Present charts 7 and 8.

Isolate the whole and the quadrilateral fractions. Identify the shape of the whole. Find the others that have the same shape. Identify the shape of the half. find the other having the same shape. Note that the shapes alternate: square (1), rectangle (1/2), square (1/4), rectangle (1/8), square (1/16).

Isolate the whole and the triangular fractions. Recall the shape of the whole. Identify the shape of the half and the others in order. Classify each triangle. All are right-angled isosceles triangles. Present charts 3 and 4. demonstrate the relationship between the lines of the figures using the arrangement pictured. As always, begin with the whole, and add one piece at a time.

· The side of the square is equal to one of the equal legs of the isosceles triangle (1/2).
· The equal side of the 1/2 triangle is equal to the hypotenuse of the 1/4 triangle.
· The equal side of the 1/4 triangle is equal to the hypotenuse of the 1/8 triangle.
· The equal side of the 1/8 triangle is equal to the hypotenuse of the 1/16 triangle.

Exercises: The child may show that the patterns demonstrated here go on infinitely, with regards to subdivisions, the shapes formed (alternate squares and rectangles), and the relationship between lines in the triangles. The child may make copies of the geometry charts.

Presentation of the Material - Triangles

Materials:
Insets of the fractional parts of a triangle
Geometry charts... triangle 1,2,3,4

Presentation: Bring out the four insets. Identify the whole triangle: equilateral, whole, unit. Identify the others in the same way as before: the number of equal parts, how they were formed, the value of each resulting piece.

The same triangle was divided into two equal parts by the height. Each has the value of 1/2. the same triangle was divided into three equal parts by the angle bisectors extending to the point of intersection. Each has the value of 1/3. the same triangle was divided into four equal parts by finding the midpoints of the sides and joining them. Each has the value of 1/4.

Classify the triangles which have been formed: 1/2 - right-angled scalene triangle; 1/3 - obtuse-angled isosceles triangle; 1/4 - equilateral triangle. show chart 1.

Examine the relationship between the lines of the fractional triangles and the whole. Show charts 2,3 and 4.

Exercise: the child may construct his own charts.

Presentation of the Concepts

1. Congruency

Materials: Metal insets of the square and its subdivisions

Presentation: Invite the child to bring out the material and arrange it properly. Take any two inset pieces from the same frame.

Note: Montessori suggests the fourths to be taken from the frame as pictured. .

Superimpose the two pieces. Indicating the surface, tell the child that every point of one figure corresponds to a point on the other figure, that is, they can be superimposed perfectly. (The Norwegians say that one "answers" the other, rather than on "corresponds" to the other) This is a

special quality that these two figures have. We can say that one is congruent to the other.
Repeat the experience with the other fractional divisions.

A. Similarity

Materials:
Metal insets of the square and its subdivisions
A red cardboard rectangle 20 x 2.5 cm
Metal insets of the triangles
Geometry charts: S9, T5

Presentation: Discuss the meaning of the word similar and give examples of things resembling one another, having the same traits, but different in other ways (similar).

Isolate the whole inset and the inset of the square 1/4. Ask the child to identify their shapes. They have the same name. are they congruent? Superimpose to find out. Hold up the small square while the child walks away with the large square to see that they look alike.

These two figures are similar. All squares are similar simply by their definition.

Isolate the 1/2 and 1/8 pieces and identify their shapes as rectangles. They have the same name. Recall the nomenclature - sides, base, height, angles. superimpose the angles to show that the angles are equal. Place two small rectangles adjacent to the larger rectangle to show that the base of the larger is twice the base of the smaller. Repeat the experiences in reference to the height. Thus, these two rectangles are similar to each other because their angles are respectively equal and the sides are in proportion to one another: the base and height of the larger are double those of the smaller.

Here the name was not sufficient to determine similarity, nor are equal angles sufficient; the sides must be proportionate.
Repeat the experience with two triangular fractional pieces: 1/2, 1/8. Classify them. Superimpose the angles. Use an extra 1/8 to demonstrate that the sides are proportionate.

Note: All square by their definition are similar to each other. Rectangles however must have proportionate sides. Compare the cardboard rectangle - 20 x 2.5cm to the 1/2 rectangle inset to see the extreme case of two non-similar rectangles. The base of the metal rectangle is twice the base of the other, while the height of the metal rectangle is half the height of the other.

(Triangle) Isolate the 1/2 triangle inset and classify it. Draw another right-angled scalene triangle which is not similar to show that the classification is not enough to render them similar. bring out a 1/3 inset and classify it. These two triangles (1/2 and 1/3) have nothing in common.

Show that the small triangle (1/4) which has the same name as the whole triangle (and thus, has equal angles) also has proportionate sides. Like the square which is the perfect quadrilateral, the equilateral triangle, which is the perfect triangle, is similar to all other equilateral triangles by its definition.

B. Equivalence

Materials:
Metal insets of the square and its subdivisions
Geometry chart: square 10 (5 and 6 for review)

Presentation: Isolate the whole and the rectangle halves and triangle halves. Recall the value of each inset. Remove the whole inset from its frame and try to place the two equal rectangles in its frame; these two squares are equal. Repeat the procedure for the triangle pieces. Refer to chart S5.

All three squares are equal to one another.

Identify 1/2 of the square; a rectangle; also a triangle. Isolate one rectangle and one triangle. Each piece has the value of 1/2 of the same square. Are they congruent? Are they similar?

When two figures do not have the same shape, but have the same fractional value; they are equivalent (equivalent: Latin *aequus* - equal; *valere*, to be worth). Repeat the experience with the fourths, eighths, and sixteenths.

If two figures have the same fractional value of the same whole, then one can be transformed into the other. Use the frame to transform the 1/2 rectangle into a 1/2 triangle. fill the space vacated by the rectangle with small pieces: 1/4 square, 2/8 triangles. Remove these pieces and arrange them on the table in the shape of the triangle.

Change the triangle into the rectangle, using the same smaller pieces and reversing the process.

Exercises:
1. For each of the three concepts, ask the child to identify the insets that bear that relationship.
2. The teacher chooses an inset piece and asks the child to find a piece which is congruent (similar or equivalent) to it.
3. The child constructs his own geometry charts.
4. The child makes different shapes (). To find the value of the pine tree, place the pieces in an empty square frame. 1/2 + 1/4 + 1/8 + 1/16 = What will make the square complete? 1/16. Thus this shape is equivalent to 1/16 less than the whole: 15/16. Later, when the child has done addition with fractions having unlike denominators, the calculation can be done arithmetically.

Note: The village is formed with triangles and squares. Therefore the value of the village is 15/16 + 15/16 = 30/16 = 1 7/8. Each roof is equivalent to one whole.

Age: Nine years

Direct Aim: To furnish the fundamental concepts: congruency, similarity, equivalence.

Indirect Aim: To serve as a base for the following material - constructive triangles.

Concepts in Action

Introduction: After the concepts have been introduced, we proceed to more advanced work in the study of equivalences, and the study of the relative value between geometric figures. For this work the constructive triangles are used.

Materials:
Three wooden boxes: One triangular, two hexagonal
The triangular box contains ten pieces; the fundamental piece is the right-angled scalene triangle, 1/2 of the equilateral triangle.
The larger hexagonal box contains eleven pieces; the fundamental piece is the obtuse-angled isosceles triangle, 1/3 of the equilateral triangle.
The smaller hexagonal box contains eighteen pieces; the fundamental piece is the smaller equilateral triangle, 1/4 of the large equilateral triangle of the first box.

Other Materials: Colored paper, scissors, glue
Right-angled ruler, graph paper, compass
Inset of the equilateral trapezoid from
"Additional insets"

Note: These materials are presented at two different levels, the first of which can be done in the Children's House as a sensorial exercise: One box is presented at a time. The child empties its contents, sorts the pieces by color and shape and unites the pieces along their black lines. The figures formed may be named at this level. Prior to the second level presentation in the elementary, make the box available to the children so that they may repeat the sensorial exercise and reacquaint themselves with the material.

A. Triangular box

Presentation: The child empties the box, sorts the pieces and unites them as usual. They all make equilateral triangles. superimpose each equilateral triangle formed of several pieces on the grey unit triangle. Determine that the unit has been divided into 2, 3, 4 equal pieces and assign the fraction values to each piece. Determine how the unit triangle was divided in each case.

Isolate the two right-angled scalene triangles. As the child did with the blue triangles of the first series, he tries to form all of the figures possible with these two triangles. Excluding the original equilateral triangle, there are five: rectangle, two different parallelograms, obtuse-angled isosceles triangle, and a kite. The child names the figures as he makes them.

Determine equivalence between each figure and the unit triangle. superimpose the two triangles on the unit triangle to show congruency, and to recall their value. Any other figure made with these two halves has the same value as the unit, and is equivalent. Show that the figures are equivalent among themselves.

rectangle = equilateral triangle
> therefore: rectangle = kite
kite = equilateral triangle

As a parallel exercise, the children may trace the triangles onto colored paper, cut them out, construct the figures using a compass and ruler.

Next, we draw the child's attention to the relationships that exist between the lines of the equilateral triangle and those of the five figures. Then we examine the relationship between the lines of the five figures.

Recall the nomenclature of the equilateral triangle: side, base, height (shown by superimposing one-half) and semi-base. Be careful to position the parallelograms so that the height is represented by a side of the triangles which form it. Identify the properties of the kite.

Rectangle: base = 1/2 the base of the equilateral triangle
height = the height of the triangle
Parallelogram: base = the height of the triangle
height = 1/2 the base
side = side
Kite: longer side = height
shorter side = 1/2 the base
long diagonal = side

Note: In identifying the nomenclature of this figure, the child showed where the shorter diagonal would be. The characteristic of this figure is that one diagonal is the perpendicular bisector of the other. However, there is no relationship to be made with the short diagonal. Later the child may discover that this short diagonal is equal to the side of T2.

Obtuse angled isosceles triangle: base = twice the height
height = 1/2 the base
equal sides = sides

In examining the relationship between the lines of the various figures, paper figures must be constructed for each figure in order to leave the triangle free for use as a measuring instrument. Be sure that the child understands that the number of comparisons to be made will decrease with each figure: the first is compared with four others, the second with three, the third with two and the fourth with only one, the last figure.

Arrange the figures so that the key figure is isolated above the row of other figures. Place the triangle first on the key figure and name a line, then find the corresponding line on the figure below. Identify all of the relationships between the key figure and the first figure, then go on to do the same with the other figures in the row. After the possibilities of the first key figure have been exhausted, one of the figures below becomes the key figure, and so on.

B. Large hexagonal box

Presentation: Empty the contents of the box. Sorting the pieces by their shape, the child observes that all but one of the pieces are congruent obtuse-angled isosceles triangles. Sorting them by color and uniting them as always, the child identifies the four figures: hexagon, triangle, rhombus, parallelogram.

Isolate the figures in yellow. Turn the three triangles of the hexagon over as if on hinges so that they are superimposed wrong side up on the unit triangle. Thus the three outside triangles are congruent to the unit triangle, and the hexagon is equal to twice the equilateral formed by the three yellow triangles nearby.

Bring back the other pairs of triangles and determine the value of each figure in relation to the unit triangle. Invite the child to experiment to try to find another figure - the arrowhead. determine its value. conclude: since the rhombus = 2/3 unit triangle, and the arrowhead = 2/3 unit triangle, then, rhombus = the parallelogram = the arrowhead.

Invite the child to experiment with three obtuse- angled isosceles triangles to make all of the possible figures. There are four: two concave pentagons, and obtuse-angled trapezoid and an isosceles trapezoid. The value of each of these figures is 3/3, thus they are equivalent to the unit triangle and equivalent to each other. Again the child forms the various figures by tracing, cutting, pasting or drawing.

As before, the lines of the various figures are compared first with the guide triangle, then among themselves. the same is done with figures formed of three triangles.

Form the hexagon again, observing that the inscribed figure is 1/2 the circumscribing figure. The three black lines of the hexagon are special diagonals (recall how diagonals are formed). These connect a vertex with the first non-successive vertex it meets.

At this point we examine the equivalence between one figure (hexagon) and the sum of several congruent figures (rhombi). Isolate the hexagon and the triangle (the figures formed by the yellow pieces). Open the hatch of the hexagon and remove the inscribed triangle, replacing it with the triangle made of three parts, and reclose the hatch. Notice that the hexagon still has the same value, though it was made of four pieces and is now made of six. Divide the hexagon into three rhombi to show that this one figure is equivalent to the sum of these three congruent figures.

As usual, examine the relationship between the lines of the hexagon and the equilateral triangle; the lines of the hexagon and those of the rhombus; the lines of the rhombus and the equilateral triangle (the diagonals are the only lines considered).

Note: This first hexagon will be called H1. The unit triangle is congruent to that of the first box and therefore is called T1. Here we have found that T1 = 1/2 H1 and T1 is inscribed in H1.

C. Smaller hexagonal box

Presentation: At first the large yellow equilateral triangle and the six obtuse-angled isosceles triangles are removed from the box. The remaining triangles are all small equilateral triangles: six grey, three green, two red.

The child begins as usual. He names the figures he has formed: hexagon, trapezoid, rhombus. Then values are assigned to the three figures. Superimpose all of the triangles to show congruency. Reconstruct each figure and count the number of congruent triangles in each. The trapezoid has 1/2 as many pieces as the hexagon. Separate the hexagon to show two trapezoids. The rhombus has 1/3 as many pieces as the hexagon. Separate the hexagon into three rhombi. comparing the trapezoid to the rhombus, we see that the rhombus is 2/3 of the trapezoid, or the trapezoid is 3/2 of the rhombus.

Examine the relationship between the lines of the three figures. Note this time that the diagonals of the hexagon connect opposite vertices. classify the trapezoid. It is an isosceles trapezoid, but it is more than isosceles. Since it is made up of three equilateral triangles, it has an extraordinary characteristic. The longer base is equal to the equal legs. Present the inset and the label and add it to the other insets. Just as the equilateral triangle is an isosceles triangle plus, the equilateral trapezoid is an isosceles trapezoid plus. show also that bilateral symmetry exists in the hexagon and rhombus.

At this point the large yellow equilateral triangle and the six red obtuse-angled triangles are returned. When the child joins the triangles, three rhombi are formed. The triangles are stacked up to prove that they are congruent. Thus the three rhombi are congruent.

Put the three rhombi together to form a hexagon, thus the hexagon is formed of six equal triangles. Open the hatch of the hexagon and take out the three red triangles, replacing them with the yellow equilateral triangle. Observe that the equilateral triangle is inscribed in the hexagon. Superimpose the red triangles (that were just removed) on the yellow triangle to show that the equilateral triangle is made up of three red triangles. Since the hexagon is made up of six red triangles, the triangle is 1/2 of the hexagon.

Note: This hexagon will be called H2 and the large yellow equilateral triangle is called T2. Therefore T2 = 1/2 H2 and T2 is inscribed in H2. The smaller equilateral triangles which are congruent to the small equilateral triangles of the first box, and therefore have the value of 1/4 T1, will be called T3.

D. Relationship between T1 and T2

Presentation: With the pieces of the small hexagonal box, form various figure: hexagon, trapezoid, rhombus, and equilateral triangle and hexagon. Superimpose the grey hexagon on the red and yellow hexagon to demonstrate congruency.

Recall the value relationships already established. Since the equilateral triangle is equal to 1/2 the red hexagon, then it is also equal to 1/2 of the grey hexagon - or a trapezoid.

Bring from the triangular box the grey unit triangle (T1) and the four small red equilateral triangles (T3).

We know that the yellow triangle T2 is equivalent to the trapezoid which is 1/2 of the grey hexagon.

One of the triangles of the trapezoid is congruent to one of the small red equilateral triangles (T3). Superimpose them. Thus, one red triangle is 1/3 of the trapezoid.

The red triangle as we know is 1/4 of the large grey unit triangle (T1). Four of these red triangles are equivalent to the grey unit triangle.

Because the yellow triangle (T2) is made up of three of the small red triangles (T3), then we can say that the yellow triangle (T2) is made up of 3/4 (T3) of the grey unit triangle (T2 = 3/4 T1).

E. Difference and ratios between similar figures

Materials: From the triangular box : T1-grey equilateral, and T3- red equilateral
From the large hexagonal box: T1-yellow equilateral, three yellow obtuse-angled triangles having the black line on the hypotenuse
From the small hexagonal box: T2-yellow equilateral, six small grey equilaterals

Presentation: Identify the two triangles by the symbol names T1 and T2. Construct the hexagons and identify them H1 and H2.

If the child has not already discovered it, lead him to the conclusion that T1 - T2 = 1/4 T1. since T1 has the value of 4/4ths, and T2 has the value of 3/4ths, T1 - T2 is the same as saying 4/4 - 3/4 which equals 1/4. Set up the equation using the materials and card for the signs. T2 = T1 - T3.

This means that the small red equilateral triangle has the same value as the grey portion that is left showing when T2 is superimposed on T1 concentrically, or so that one vertex coincides.

Knowing that H1 is the double of T1 and that H2 is the double of T2, we can conclude that their difference would be the double of 1/4 of T1, that is 2/4.

2T1 - 2T2 = 2 (1/4 T1) = 2/4
2(4/4) - 2 (3/4) = 8/4 - 6/4 = 2/4

Using T1 as the unit, assign relative values to the hexagons: H1 = 8/4 and H2 = 6/4 (T1 = 4/4). The large hexagon is 8/4 of the grey triangle; the small hexagon is 6/4 of the grey triangle. Therefore, the small hexagon is 3/4 of the large hexagon. Examine also the inverse of each of these statements as they are also true.

Since the ratio between the small triangle and the large triangle is 3:4, then the same relationship exists between their doubles, the small hexagon and the large hexagon ... 3:4.

Superimpose H2 on H1 concentrically with the sides parallel. the frame is the difference between the two hexagons, and, therefore, has the value of 2 1/4 equilateral triangles. Thus the difference between the hexagons is a rhombus. This is the arithmetical way of showing their difference.

F. Sensorial demonstration of the difference between H1 and H2

Materials:
Triangle fraction insets: whole, 3/3
Constructive triangles: H1, H2 and two red T3

Presentation: verify the congruency between the triangle whole inset and the small red equilateral triangle, by superimposing the inset on the wooden triangle. In the same way, we can see that this metal inset is equal to 1/6 of the grey trapezoid.

Superimpose H2 on H1 so that the vertices of the grey hexagon coincide with the midpoints of the sides of the yellow hexagon. Place the 1/3 metal inset pieces over the yellow portion that remains visible. With this material we can cover only three places. However these three obtuse-angled triangles equal one small red equilateral triangle. Move the three thirds to the three remaining vacant spots; these three triangles correspond to the second small red equilateral triangle.

G. Equivalence between two rhombi

Material:
From the small hexagonal box: two red obtuse-angled and two red equilateral triangles
From the triangular box: grey equilateral, green right- angled
2/2 triangle fraction insets (optional)

Presentation: Join each pair of red triangles along the black lines to form two rhombi. Superimpose one rhombus on the other to show congruency. Demonstrate that each rhombus was divided into two equal pieces. Since each triangle has the value of 1/2 of the same rhombus, all of the triangles are equivalent.

Demonstrate this equivalence sensorially by superimposing the two metal inset pieces on first one red triangle, and then in another arrangement on the other.

We know that the red equilateral triangle has the value of 1/4 T1. Since the red obtuse-angled triangle is equivalent, it must also have the value of 1/4. Therefore, 1/4 + 1/4 = 1/2 T1. Superimpose the green right-angled triangle as a proof.

Following this line, the grey triangle can be constructed with two rhombi. Also this 1/2 is the difference between the two hexagons in the form of a rhombus or this right-angled scalene triangle.

H. Equivalence between the trapezoid and T2

Materials:
From the triangular box: grey equilateral triangle
From the small hexagonal box: yellow equilateral, three green equilateral triangles

Presentation: Unite the three green triangles to form a trapezoid. Knowing that each of these small triangles is congruent to the red triangle having the value of 1/4 T1, we can say that the trapezoid has the value of 3/4 T1. The trapezoid can be superimposed on the grey equilateral triangle to show that 1/4 is lacking.

We've already seen that the red obtuse-angled triangle was equivalent to the small red equilateral triangle having the value of 1/4 T1. So each green triangle would also be equivalent to a red obtuse-angled triangle. These obtuse-angled triangles were each 1/3 of T2; T2 is composed of three obtuse-angled triangles having the value of 1/4 T1. Therefore T2 is equal to 3/4 T1. Having the same value of 3/4 T1, the trapezoid and T2 are equivalent.

I. Ratio between circumscribing and inscribed figures

Materials:
From triangular box: grey equilateral, four red equilateral triangles
From large hexagonal box: yellow equilateral, three yellow obtuse-angled
From small hexagonal box: six grey equilateral triangles
Triangle fraction insets 4/4, square fraction insets 4/4 (diagonal)

Presentation: Superimpose the four small red equilateral triangles on the grey equilateral triangle. Remove the three triangles at the vertices, leaving two equilateral triangles.

An equilateral triangle inscribed in another equilateral triangle is 1/4 of it. The ratio is 1:4. Demonstrate the same experience using the metal insets. We can construct a chart to examine all regular polygons.

Number of sides Ratio

3 1/4
4 2/4
5 ? (between 2/4 and 3/4
6 3/4
etc... etc...
circle 4/4

Use the metal inset of the square. Recall the value of the triangular pieces. Put two pieces aside. With the remaining two form an inscribed square. The inscribed square is 2/4 of the circumscribing square.

We don't have any material to examine the pentagon. For the hexagon, form H1 and H2 and superimpose them to recall the ratio of 3:4.

We have no more materials to examine the others. When will the ratio be 4/4? When there is no difference between the circumscribing and inscribing figures, that is when the figures are circles. We can conclude that the ratio for the pentagon will be somewhere between 2/4 and 3/4, and for the septagon and all the others, the ratio will be between 3/4 and 4/4.