Math :: 6-9

MONTESSORI TEACHERS COLLECTIVE (MOTEACO.ORG)

Table of Contents:

Introduction to the Decimal System

'Our aim is not only to make the child understand, and still less to force him to memorize, but so to touch his imagination as to enthuse him to his innermost core.'
- Maria Montessori

With the first mathematics materials the child is introduced to the numbers one through ten. The materials that follow will develop the concept of the decimal system, that is, a numerical system based on ten. Maria Montessori called the decimal system the "cell of our system."

To understand the decimal system is not easy for a child. It took humans many years to realize that the value of a numeral is dependent on the position it occupies. It was much later that the concept of zero was developed, and even later that the decimal point came into existence.

Our decimal system (base 10) has nine numerals, one through nine. The presence of one or more zeroes allows us to create numbers beyond nine up to infinity. Thus, learning the numbers one through nine and their numerals, in addition to the concept of zero, is the only truly difficult part for the child. This he has already accomplished. Counting experiences (adding one more) up to 10 have preceded; now the child will learn to count beyond 10.

In the hierarchical orders-ones, tens, and hundreds of the simple class; ones, tens, and hundreds of thousands, and so on- there are nine units: one through nine. No matter in which hierarchy the numeral one appears, the absolute value of one is one. The relative value depends on its position. The limit between one hierarchical order lies in the 'secret of ten' and in the exact value of the numerals one through nine. It is necessary that the child fixes in his mind the concept of the hierarchical orders and their values. The materials that follow enable the child to avoid the confusion and difficulties he may otherwise encounter.

The Great Lesson

Prepare a broadly engaging, impressionistic story of the history of Mathematics. Card materials, timelines, and captivating stories will engage the child when offered with tales of early humans and their efforts to measure and quantify their universe. Explore early symbols of numeration, the history of 'zero', and prehistoric calendars. Study the precision of the Great Pyramids of Egypt. Delve into the navigation techniques of the ancient polynesians. Pursue this information enthusiastically and your students will become enthused, as well. Present this work graciously, and your students will express grace in their own studies. Their understanding of arithmetic within the context of human progress will grow.

This understanding of how arithmetic evolved, and continues to evolve today, will inform an appropriate awe of our 'language of numbers'. Continue your 'Great Lesson' throughout the years, through attention to technology & current events and an ongoing expressed passion for the measurement of our world. This passion is a 'Fundamental Need', a common thread across cultures that ties humanity together.

Numeration

Quantities and Symbols

QUANTITIES IN THE DECIMAL SYSTEM

Materials:
...the golden bead materials which consist of:
...one container of loose gold beads representing units
...one box of gold bead bars of ten beads each
...one box of 10 gold bead squares of ten bars (representing 100)
...one box containing 1 gold bead cube of hundred- squares (representing 1,000)
...a large tray with a dish or smaller tray, used for transferring the quantities

Presentation:
Individual Presentation. As a unit bead, and then a ten bar is placed on the table, the child is asked to identify the quantities. One hundred and one thousand are presented also. The teacher gives a three period lesson naming the quantities: unit, ten, hundred, and thousand. The child is then invited to examine the materials and their composition. The child may count the ten beads on the ten-bar again. "The hundred is made up of ten ten bars". The ten-bar is placed on top of the square as the child counts. "The thousand is made up of 10 hundreds". The hundred-square is placed next to each section of the cube as the child counts. The teacher gives the three period lesson defining the composition of the quantities.

Exercise:
Small Group exercise. The golden bead materials, now including the wooden hundred-squares and thousand-cubes are arranged at random on a rug (in a basket). Each child takes a tray. The teacher asks the child to bring a quantity. 'Bring me 3 hundreds' As each child returns with the quantity, the child identifies it, and the teacher and child count it together. At first the child is asked to bring only one hierarchy at a time. Later he will bring all four at once.

Age: 3-6

Direct Aim:
...to develop the concept of the hierarchical orders of the decimal system: units, tens, hundreds, thousands.
...to give the child the relative measurement of the quantities: bead, bar, square, cube.

Indirect Aim:
...to prepare the child for geometry concepts: point, line, surface and solid.

NUMERALS (SYMBOLS)

Materials:
decimal system numeral cards:

...1-9 printed in green
...10, 20...90 printed in blue on double-sized cards
...100, 200...900 printed in red on triple-sized cards
...1000, 2000...9000 printed in green on quadruple-sized cards

Presentation: 1st Part
Individual Presentation. As the one and the ten cards are placed on the table, the child reads them. One hundred and one thousand are presented in a three period lesson. The cards are arranged as in the diagram. Then the child examines the particular characteristics of each numeral, its color and the number of zeros.

Games:
1. The cards are turned face down on the table. Without turning the card face up, the child identifies the numeral indicated by the teacher. How many zeros does it have? The card is turned up to control. Another time, the teacher asks the color of each numeral.
2. 'Magician'. The teacher picks up the four cards arranging them in a pile weighted to the left. This arrangement is shown to the child. The cards are stood on end as the top cards slide into the second position. Where did all the zeros go? They seem to have disappeared, but they are still there. The cards are lifted one by one to reveal the zeros. The child performs the magic trick.

Presentation: 2nd Part:
The first four numeral cards, just previously presented, are lain in order. The remaining unit cards are placed in a column below one, the child being encouraged to read each as he lays it in position. This continues for the tens (one ten, two tens...), hundreds (one hundred, two hundred...), and thousands (one thousand, two thousand...). The three period lesson continues noting color and number of zeros as well. If the child is familiar with the names, twenty, thirty..., these may be supplemented. It is important for the child to realize that twenty (20) is two tens.

Age: 3-6

Direct Aims:
...to understand the orders of the decimal system.
...to turn the numerals for each of those four orders

Indirect Aim: to understand the importance of zeros in distinguishing the numerals.

UNION OF QUANTITIES AND NUMERALS (SYMBOLS)

Materials:
...golden bead materials
...numeral cards 1-9, 10-90, 100-900 and 1000

Presentation:
As the teacher lays out the unit beads, the child counts: 'one unit, two units...nine units.' The teacher goes on: 'If we added one more unit, we'd have ten units. Ten units make one ten.' The tens are counted as they are lain out: 'one ten, two tens... nine tens.' 'If we add one more ten we'd have ten tens. Ten tens make one hundred.' And so on up tone thousand. Here the rule of the decimal system is stated: Only nine quantities can remain loose. When we reach ten, we move to a superior hierarchical order.

Exercises:
1. The teacher places the numeral cards (as in the diagram) on one table and the quantities on another. The teacher places one quantity on a tray. The child finds the corresponding numeral card

and places it on top of the quantity. The teacher controls.
2. The teacher places a numeral card on a tray. The child brings the corresponding quantity.

Subsequent Presentation:
Group Presentation: The teacher places cards of different orders on the tray. The child brings the corresponding quantities with the cards placed on top. The teacher controls and hands the cards back to the child. When the child has all of the numeral cards, he does the magic (arranges the cards) and reads the numeral. The exercise continues omitting one hierarchical order to show that the place is held by zeros.

Age: 3-6

Direct Aim:
...to understand the rule of the decimal system: only nine quantities can remain loose.
...to familiarize the child with the hierarchical orders
...to offer the opportunity to write complete numerals

Indirect Aim:
...to give the understanding that zero occupies the place of a missing order.

Note: With these and all other activities involving the golden bead material, the units should remain in the small tray. This confines the loose beads in a set and makes it easier for the child to see that he has nine, one more would make ten. When counting, the beads may be dumped into the palm and counted back into the tray.

ADDITIONAL EXERCISES IN NUMERATION

The Hundred Board

ADDITIONAL EXERCISES IN NUMERATION

The Seguin Boards

Teen Boards

Materials:
...box containing two boards and 9 wooden tablets for 1-9
...box of ten golden ten bars
...box of 1 each of colored bead bars 1-9

Presentation:
Individual presentation. The teacher presents the boards side by side and the tablets ordered in a row. Indicating the first slot, the child reads the numeral 10 and places a ten-bar to the left of that slot. The teacher then adds a unit bead and the tablet - 1 to make eleven. 'This numeral is eleven: eleven is ten and one.' This continues through nineteen. When counting the beads the child counts ' ten, eleven, twelve... ten and two is twelve.' Three period lesson follows naming the quantities and in the second period forming them.
If the child questions why the last slot is blank, explain that in order to make the numeral that comes after nineteen, other materials are needed.

Age: 3-6

Aims:
...to clarify understanding of the decimal system (11 means: 1 ten and 1 unit)
...to progress in counting from 10 up to 19
...to learn the names of numbers 11-19

ADDITIONAL EXERCISE IN NUMERATION

The Seguin Boards

Ten Boards

Materials:
...box containing two boards with numerals 10, 20, 30....90, and 9 wooden tablets for 1-9
...box of 9 gold unit beads
...box of 45 gold ten-bars
...1 golden hundred square

Presentation:
Individual presentation. With these materials we will be able to make the numeral that was missing from the teen boards.
a) Only the boards and ten-bars are used for now. Pointing to the first numeral 10, the child is asked to identify it and place the correct quantity next to it. The child identifies the next numeral 20 as two tens. We call this twenty. The ten-bars are placed next to twenty, and counted 'ten, twenty.' This continues, identifying numbers by correct names and counting the ten-bars by 10's. Now we have counted by tens up to ninety. The three period lesson follows.
b) The ten-bars have been returned to their box. Again the child identifies 10 and brings out one ten-bar. After ten is eleven: the one tablet is placed in the slot and one unit bead is added 'ten, eleven.' This continues up to nineteen. After nineteen is twenty: Twenty is two tens, so we put away the nine unit beads and take another ten-bar. Both ten-bars are moved down by twenty. This one-by-one counting continues up to 99. If we added one more bead, we'd have 10 units which make another ten-bar. Then we'd have ten ten-bars which makes one hundred. After 99 comes 100. The hundred square is placed next to the blank space.

Age: 3-6

Aims:
...to clarify understanding of the decimal system (11 means 1 ten and 1 unit)
...to count from 1 to 99
...to learn the names of numbers 20-99

Note: These materials may be presented any time after the Union of Quantities and Numerals of the Decimal System.

Introduction to Operations Using the Change Game

STATIC OPERATIONS IN THE DECIMAL SYSTEM

Materials:
...golden bead materials including wooden hundred squares and thousand cubes
...large numeral cards
...three sets of small numeral cards
...a box containing symbols for operations +, -, x,÷

...small pieces of paper
...a thin rod to be used for the = line
...a soft cloth.

a. **Presentation of Addition**:
Small Group Presentation. Each of two or three children takes a tray. The teacher states a different numeral for each and they find the appropriate small numeral cards and the quantity, placing the cards on top of the respective quantity. The teacher controls. The child arranges the cards, places the numeral on the table and dumps the quantity on the cloth. When all the quantities are on the cloth, the teacher gathers up the cloth, mixing all the quantities together. The cloth is opened and the materials are sorted. The child begins with units counting the quantity and bringing the large numeral card. When all has been counted, the child arranges the cards and reads the quantity that the combination has produced. Pointing to small numeral cards: 'The children brought these small quantities. When we put them together we made this large quantity." (indicating the large numeral cards which is seperated from the addends by the thin rod) 'We have done addition.'
The numerals are arranged in a column. The plus sign and its function is presented. The line (which was formed by the thin rod) is equivalent to the = sign. The teacher reads the problem (equation) '2,512 plus 1,234 equals 3,746.'

b. **Presentation of Subtraction**:
Group Presentation: Initially the teacher may play the "Rich Man, Poor Man" game to demonstrate the concept of "taking away." The teacher has a large quantity from which several children take away small quantities until there is nothing left. The purpose of this game is to make the impression of taking away and nothing remaining.
The child has an empty tray. The teacher has a large quantity on his tray. The quantity is counted beginning with the units and large numeral cards are placed on the quantities. The child arranges these cards and reads the numeral. Offering the child some of this large quantity, the teacher chooses some small numeral cards. The child arranges these cards and reads what shall be taken away. The teacher counts out this quantity from what is on the tray, beginning with units. What is left? This quantity is counted and small numeral cards placed on the quantities, arranged and read. What remains on the tray is the result of subtraction. When we take away, we are subtracting. The problem is set up with the minus sign and read. The large cards tell us the large quantity; the smaller cards are for the small quantity that was taken away and the small quantity that remains.

c. **Presentation of Multiplication**:
Group Presentation: Each child is given a tray and is asked to get the cards and quantities for a stated number. The teacher controls each child's tray; the cards are arranged, the numeral is read and the quantity is placed on the table. As in addition the quantities are put together, sorted, counted, labeled and the sum is read. The problem is then set up as in addition with the plus sign. Now it is observed that in this 'special' addition, all of the quantities put together (addends) are the same. This special addition is called multiplication. Taking one small numeral : 'We can say that we took this quantity three times.' The times sign is presented and the numeral three is written on a blank piece of paper. The result has not changed; this is just an easier way to write the problem.

Note: After this initial presentation, the child no longer sets up the addition problem first.

d. **Presentation of Division**:
Group Presentation: The children are seated in a circle. One child is asked to pick up the large numeral cards for the stated quantity, and he brings the golden bead material. 'This large quantity must be distributed to each of these other children equally. 'Starting with the thousands, one thousand for you, one thousand for you, another thousand for, another thousand for you'... until all of the quantity has been distributed. The children who received count their quantity to be sure that everyone received the same amount. One child is asked to get the small numeral cards. It is

emphasized that each child received this amount. When we distribute equally to many others, we divide. The division problem is set up, using a small piece of paper for the divisor, and it is read. The result of division is what one child receives.

Exercises:
After each problem has been demonstrated and set up with numeral cards and symbols, the child may write this in his notebook, preferably on paper with columns and in colors for the hierarchical orders.
After all of the operations have been presented, it is important for the child to understand the function of each operation. 'What is addition?... putting together...etc.

Age: 3-7

Control of Error: The teacher checks the quantities counted.

Aims:
...to realize the concept of addition (putting together), subtraction (taking away), multiplication (adding the same number many times), and division (distributing equally)

DYNAMIC OPERATIONS IN THE DECIMAL SYSTEM

Materials:
...golden bead material
...large and small numeral cards
...symbol cards for the operations
...problem cards for each operation

a. **Introduction to the Change Game**:
Individual Presentation. A large quantity is placed on the tray and the child is invited to count it. Beginning with units, the child counts, but is stopped at 10. Ten units cannot remain loose; they must be changed for a ten-bar. The ten beads are traded for one ten-bar from the bank. The child continues counting units and placing the correct large numeral cards on the try. So on to thousands. The cards are arranged and read. The child does many exercises.
Aim: to exchange equal quantities of different hierarchies
to reinforce the rule: only 9 units can remain loose
to reinforce knowledge of the composition of each hierarchy (ten tens=100)

b. **Presentation of Addition**:
The teacher reads a task card. The child performs each command as it is read. The teacher controls.

c. **Presentation of Subtraction**:
The teacher reads a task card. The child performs each command as it is read. The teacher controls.
The teacher presents the thousand cube (golden bead) and wants to take away 1 unit. This may be symbolized with the large and small numeral cards for emphasis. How can this be done? The thousand is changed to 10 hundreds. Now can we take away one unit? Not yet. So on until one unit can be taken away. The remaining quantity is counted and represented with small cards.
Aim: to realize that one unit revolutionizes a large quantity.

d. **Presentation of Multiplication**:
As for addition task cards are prepared.

e. **Presentation of Division**:
Group Presentation. As with static division the child sets about distributing. When he finds that he

doesn't have enough for one hierarchy to go around, he must exchange for a lesser hierarchy. When there is a remainder, the corresponding small numeral cards are brought and placed after a small card with the initial r to the right of the result (quotient)

Age: 4-7

Aim:
...to further understand the concept of addition, subtraction, multiplication, and division

Golden Bead Chains

CHAIN OF ONE HUNDRED

Materials:
...a chain formed of 10 ten-bars
...a hundred square
...an envelope containing: 9 units arrows 1-9 in green, 9 tens arrows 10-90 in blue, a red hundred arrow

Presentation:
The chain is folded like a fan to resemble a hundred square. Do you recognize this? It looks like 100. We prove that it is 100 by placing the hundred square on or beside the folded chain. The chain is stretched out to its full length. How many tens are there in this hundred square? How many tens are in this chain? The square and the chain are exactly equal.
The child begins counting the beads placing the corresponding arrows by the bead. At 10, he begins counting by tens to 100. The red hundreds arrow and the hundred square are placed by the last bead.

Exercise:
1) The unit arrows are removed and the tens arrows are turned over. The child counts by 10's to 100, and then backwards by 10's.
2) The teacher asks the child to indicate a number on the chain. Then pointing to a bead, asks, 'What is this?'

Age: 3-6

Aims:
...to represent one hundred in a line
...to learn numeration from 1-100
...to count forwards and backwards by 10, s from 1-100

CHAIN OF ONE THOUSAND

Materials:
...a chain of 100 ten-bars with a ring after every 100 beads
...an envelope containing: 9 green units arrows, 9 blue tens arrows 10-90, 9 red hundreds arrows, and 1 green thousands arrow
...10 hundred squares
...1 thousand cube

Presentation:
The chain is stretched out to show the difference between this chain and the chain of 100. It is

folded like a fan to resemble hundred squares. It is proven that there are ten hundreds in this chain by placing the hundred squares on top of each section. The hundred squares are then stacked up to prove that this chain is equal to the cube. After this correspondence has been firmly established, the child begins counting, first by units, matching the arrows, then by tens, and lastly by 100's. At each hundred the child places a hundred square. At 900 the child counts by 10's again to 990. The child counts by units from 990 to 1000. We place another hundred square here, but now we have 10 hundreds. Ten hundreds make one thousand, so we can put the cube here instead.

Exercise:
1) The child counts by 100's to 100, forwards and backwards from 1 to 1000, with the arrows overturned
2) The teacher asks the child to point to a number on the chain. Then pointing to a bead, the teacher asks, 'What is this?'

Age: 3-6

Direct Aim:
...to count forwards and backwards by 10, s and 100's to 1000

Indirect Aim:
...to prepare for learning the powers of numbers

Hierarchical Material

INTRODUCTION

These materials are the geometric representation of the quantities from one unit to one million-the powers of ten; 100 to 106 Having reached one million the child will easily imagine the succeeding hierarchies.

Materials:
...the wooden materials made of light wood to facilitate movement, in relative proportions:
...1 - green cube - .5cm
...10 - blue rod with green lines .5 x .5 x 5cm
...100 - red square with blue lines .5 x 5 x 5cm
...1000 - green cube with red lines 5cm
...10,000 - blue rod with green lines 5 x 5 x 50cm
...100,000 - red square with blue lines 5 x 50 x 50cm
...1,000,000 - green cube with red lines 50cm
...numeral cards 1; 10; 100; 1,000; 10,000; 100,000; 1,000,000 all on white backing with numerals printed in black
...a ruler or stick or an expensive laser pen

Presentation:
The materials should be laid out in a row as they are presented, from right to left. Isolate the unit cube and identify it. This is one. If I had ten of these little cubes and placed them end to end, I would have this rod. This is ten. Place the cube along the side of the rod to count the ten sections. If I had 10 of these tens, I would get one hundred-this square. Count the sections of the square using the rod. Place the three pieces on the table in a row, and place the ruler on top. These all have the same height. Identify the three pieces again - 1, 10, 100, unit, ten, hundred. They are numbers of the simple class. Set the stick aside.
Isolate the thousand cube. This is still a unit, but it is a unit of the thousands. Compare its color and shape to the unit cube. Present the ten and hundred as before. Place the stick on top to see that

they are all the same height. They are 1, 10, 100 but of thousands.
Present the million cube. This is still a unit, but it is a unit of millions. Imagine the ten of millions. It would be as long as ten of these side by side. Imagine also the hundred, which would be made of ten of these tens. These would make up the class of millions.
Review the first period giving the names of the classes and the names 1, 10, 100 for the cube, rod, square, to show how these three orders are repeated in each class. The dominant figure of each class is the cube for it gives us the name of the class. Compare these materials to the concepts of point, line, surface and solid-which is only a point of the next line. The point is represented bigger each time. Even the Earth, as big as it is, is just a tiny point in space. Three period lesson

Give the child the symbols to match by placing on top of the material. Identify for the child the symbols of 10,000 · 100,000 · and 1,000,000. Notice that the comma corresponds to a change in hierarchical class.

Games:
Distribute the cards to a group of children and they place the cards on the appropriate material. Give each child a piece of material and he finds the right card.
Ask the child to identify a piece of material, the class to which it belongs, the reason for the color of the lines, of what it is composed. In this way the child will be able to form definitions in his own words. Emphasize that the superior hierarchy is always formed of 10 of the preceding hierarchy.
The child draws the material in his notebook or cuts and pastes the pieces using a different scale of measurement. The cube is drawn as a three-dimensional image.

Age: 7 years

Introduction to Other Mathematical Materials

STAMP GAME

Materials:
...wooden stamps of four types:
...green unit stamps printed with the numeral 1,
...blue tens stamps printed with the numeral 10,
...red hundred stamps printed with the numeral 100, and
...green thousand stamps printed with the numeral 1000
...box with three compartments each containing 9 skittles and one counter in the hierarchic colors;
...four small plates

Presentation:
One of each stamp is presented and identified, lain in correct order-units t the far right; thousands to the left. The teacher forms a number laying out the stamps in a straight column for each hierarchy. 'Can you read this to me?' Now the child reads a number from a slip of paper and forms the number with the stamps. After the child has done many exercises of this type, he will be ready to go on to operations.

HIERARCHICAL BEAD FRAMES

INTRODUCTION

In this work, which follows memorization, the child encounters a new difficulty. He must identify quickly the value of each digit of a number as it is indicated by the place the digit occupies. The child considers the position of the digits in a number, and determines the value of each digit according to

this position.
The decimal system material: the bead-1, bar-10, square-100, and cube-1000, represented constant values, values which did not change when the position of the material changed. On the bead frames beads of the same size represent the various quantities, thus eliminating the sensorial element of size. The quantities are symbolically represented on the bead frame aided by the hierarchic colors and the relative positions of the frame.
These hierarchic colors have been encountered before in the decimal system material numeral cards and the stamp game. On the frame one blue bead represents ten (unit) beads of the previous hierarchy and one-tenth of a (hundred) bead of the next superior hierarchy.

SMALL BEAD FRAMES

First Presentation:

Materials:
...small bead frame, corresponding form
...a golden unit bead, 10-bar, 100-square, 1000-cube
...2 green beads (from the unit division board)

Presentation:
Introduce the child to the concept of hierarchy with an analogy: i.e. the social organization differentiates one person from the next. The same thing happens in the beads, These 4 beads could be units of the simple class or units of the thousands depending on their position.
These three colors, green, blue, and red are repeated in each class in the same sequence. Only their position will differentiate them.
Isolate two green loose beads. How many are there? 2 On the frame isolate one unit and one thousand bead. Here I also have two green beads, but I can't call them just '2 green beads.' The one at the top has the value of one; the bead on the lower wire has the value of 1000. The position makes the difference.
The absolute value is the value of the unit independent of its position (the number of beads on the wire). The relative value is the value of a digit when its relative position is taken into consideration.

The History of The Abacus
Relate the story of the abacus: A bead frame like this is used by children all over to learn to count. It is a very, very old instrument, that was used by the Chinese as far back as 500 BC. They called it 'swan-pan'. The Japanese caught on to the idea, but they called it 'soro-ban.' The Russians learned about it and began to use it in their country, calling it 's-ciot', which means calculator. Around 1812 there were French prisoners in Russia who learned about the abacus. When he was released he brought the idea back to France. This knowledge spread rapidly around Europe and to America. Studies have shown that this design originated long, long ago. People made little grooves in the sand and placed little pebbles into the grooves. Each groove was like one of our wires, and the pebbles were like our beads.

Introduction to the materials
Our bead frame has four wires; the first three are equal distances from one another, and between the third and fourth there is a greater distance. This space separates the simple class from the class of thousands. On the right side we see these two classes indicated by two different colors. There are ten beads on each wire. The number on the left side of the frame indicate the value of each bead on that wire. Here it says 1, so each bead has the value of one... and so on to 1000.
On this form the same situation is repeated. Turn the bead frame on its side to demonstrate the corresponding colors, names of classes, and the space to divide the classes, which has been replaced by a comma.

Passage from sensorial to symbolic representation
Isolate the golden bead, and ask the child to identify its value; one unit. Isolate one green unit bead on the right side of the frame. This green bead is also one unit. Each bead on this row is a unit. Isolate the ten-bar, and ask the child to identify its value; ten. This blue bead also has the value of ten. Each one of the beads on this row is worth ten units. Continue in the same way with the square and the cube.
By means of the three period lesson: have the child match the corresponding quantities, i.e. Give me 100. The child gives the square. Now show me 100 on the frame, or pointing to a particular bead: What is this? The child names it and gets the corresponding golden bead material.
To check the child's comprehension, isolate one unit bead and one thousand bead. These two beads are both green: do they have the same value? Why?

SMALL BEAD FRAMES

Numeration Based On Position

Materials:
...small bead frame
...form for each child

Presentation:
Moving one unit bead to the right, the teacher counts one and writes the digit on the form. The numeration continues: **move a bead,** say the number, write it down. As the tenth unit bead is moved forward: I change these ten units for one ten bead forward. Write the digit 1 on the blue line. Move another ten bead forward- 2 tens and write a 2 in the column. The numeration continues in this way up to 90, changing 10 tens for 100. Finally the numeration ends at 1000. This is controlled by 28 lines on the form.
At the end fill in all of the zeros to bring into focus the passage from one hierarchy to the next by the placement of one more zero each time.
This work recalls the concept of changing from one hierarchy to another from the decimal system operations. This activity helps the child to fix the places which correspond to each hierarchy.

Activities: Formation of Numbers
1) The teacher forms a number on the frame. The child reads it.
2) The child reads a number from prepared cards and forms it on the frame.
3) The child forms any number on the frame, reads it and records it on the left side of the form used earlier for the presentation.

Note: Each time the child forms a number he will recall the formation 10.

Aim: familiarization with the bead frame
knowledge of the passage between hierarchies

LARGE BEAD FRAMES

First Presentation

Materials:
...large bead frame, bearing the same characteristics as the small frame:
...space and change of frame color to separate the classes,
...10 beads of respective hierarchic colors on each row.
...wooden hierarchic materials

Presentation:
Slide one green bead to the right and isolate the unit cube. This green bead has the same value as this cube. What was the value of this cube? ..unit of what class? the simple class. Every green bead on this row has the value of one unit. In the same way identify each row of beads using the hierarchic materials.

Exercise:
Isolate a bead and ask the child to identify the equivalent material and ask the child to isolate the corresponding bead.

LARGE BEAD FRAMES

Numeration According To Position

Materials:
...large bead frame
...corresponding long form with 55 lines

Presentation:
Move one unit bead to the right, count one and write the digit 1 in the first space of the form. Continue counting and writing. When the tenth unit bead is moved forward, 'we know that 10 units make 1 ten.' The units are moved back and one ten is moved forward. Write '1' in the tens column (without a zero) Continue in this way up to 1 million; the form will be filled up. Go back and add the zeros. Notice the passage from one hierarchy to the next as indicated by the zeros. Note that the commas correspond to the spaces between classes.

Exercises;Formation of Numbers:
1) The teacher forms a number on the bead frame. The child reads the number and writes it on the form.
2) The teacher writes a number on a piece of paper and the child reads it, forms it on the frame and writes it on the form.
3) The child creates a number on the frame and writes it on the form. The child performs addition, subtraction and multiplication (with a one-digit multiplier) on this frame. This larger frame permits the child to work with larger numbers.

HORIZONTAL GOLDEN BEAD FRAME

Materials: the frame which lies flat on the table.

It is less sensorial in that hierarchic colors and spaces between the classes have been eliminated (note: the black lines are drawn on the board beneath the wires; they will indicate where to begin the multiplication when multiplying by units, tens, hundreds or thousands.). All of the previous operations can be done with this material, but we will do the most interesting - multiplication with a two-digit multiplier.

Introduction to Memorization

Memorization is the key that will allow the child to continue in his development of the mathematical mind. Memorization can be defined as conservation in the memory along with the ability to recall experience and impressions. Oftentimes the exercises of memorization are boring to the child because the same thing is repeated over and over until he remembers. In order for our goals to be achieved, we must find ways to make memorization attractive and interesting.

Memorization must be taught along with the decimal system materials. The child has realized the concept of the decimal system: that only nine units can remain loose, and he has understood the function of each operation. Now we must learn to calculate. As soon as the child has memorized all of the possible combinations of 1-9, he will be able to calculate any complex addition. In order to enter the world of mathematics, the child must be given the opportunity to memorize.

Addition

Stamp Game Addition

DYNAMIC ADDITION

Materials:
...wooden stamps of four types:
   ...green unit stamps printed with the numeral 1,
   ...blue tens stamps printed with the numeral 10,
   ...red hundred stamps printed with the numeral 100, and
   ...green thousand stamps printed with the numeral 1000
...box with three compartments each containing 9 skittles and one counter in the hierarchic colors;
...four small plates

Presentation:
Using dynamic addition work cards, the teacher presents addition using stamps. The child forms the first addend and then forms the second addend, starting his columns well below the first addend. Now you do the addition. The child slides the rows together and begins to count, starting with units. At ten units the child must stop and change these to a ten. The units are put back and a ten is taken out. The child continues counting and changing. Now let's see what the result is. The number is read and the problem is recorded in his notebook.

STRIP BOARD (including doubles)

a. Complete List of Strip Board Materials

Materials:
...addition strip board
...pink box containing pink and blue strips
...mimeographed booklets
...box containing 81 combinations on small cards
...box containing pink tiles with 81 sums
...box of 36 pink rectangles and 36 squares
...Control Charts I-VI

STRIP BOARD (including doubles)

b. First Presentation of the Addition Strip Board

Materials:
...addition strip board and strips

Presentation:
In order to show the child how to use the materials, the teacher presents a few short exercises. A blue strip is chosen at random. The child identifies the strip. It is placed along the top row of squares. The teacher takes a pink strip. I am going to add 5 (pink) to this 7 (blue). The pink strip is lain on the board. We can see that 7 plus 5 is 12. I read 12 here at the top, pointing to the top row of numerals. The exercises continues like this.
Then the child adds keeping the first addend the same. The second addends are chosen at random order. In these exercises the blue strip remains on the board throughout.

STRIP BOARD (including doubles)

c. Addition Booklets

Materials:
...mimeographed booklets of nine pages each; each page has
...nine combinations with a common first addend
...addition strip board and strips
...control chart I (81 combinations in 9 columns)

Exercises:
The child chooses one page in the booklet. He reads the first combination 3+1=__. The first addend is 3, so the blue strip for 3 is placed on the board. The second addend is 1, so the pink strip for 1 is added. The sum is read at the top, and is written in the booklet near the equal sign. The first addend remains the same, therefore the blue strip remains on the board. The pink strip may be turned face down in its place, to remind us that we've finished with 1. The child follows the order on the form.

If the child is just writing the numbers in succession on the column, the aim of memorization is not being met. Therefore, the child may complete a page in any order. A booklet with random order problems for one number can be introduced as well.
example:

3+2=__
3+4=__
3+7=__
3+9=__
3+1=__
3+1=__
3+5=__
3+8=__
3=3=__
3+6=__

Control of error: Chart I. The child simply compares his page to a column on the chart.

STRIP BOARD (including doubles)

d. Combination Cards

Materials:
...box of combination cards
...addition strip board and strips
...paper
...Control Chart I

Exercise:
The child fishes for a combination. He reads it and writes it down on his paper 7+6=___. The first addend is 7, so the blue strip for 7 is placed on the board. The second addend is 6, so the pink strip is placed directly next to the blue strip on the board. The sum is read at the top and is written on the paper next to the equal sign. The strips are put back in their places. The child fishes for another combination, and the exercise continues.

Control of error: Control Chart I. The child looks at the first addend, finds the column where the combinations have that first addend, then looks for his combination.

STRIP BOARD (including doubles)

e. The Combinations of One Number

Materials:
...addition strip board and strips
...paper
...Control Chart I

Exercise:
Let's see all of the different ways to form 10. The blue strip for one is placed on the board. What do we need to make 10? The pink strip for 9 is added, 1+9=10. The child continues in order, making combinations until 1+9=10. The child then writes the combinations and sums on his paper.
Now the child observes that the pink strip decrease in size as the blue strips increase. Also it is observed that 9+1 is the same as saying 1+9. The 9+1 are held up to the 1+9 strips to compare. If I remember that 1+9=10, then I also remember that 9+1= 10. We can eliminate one of these combinations. The strips are put back in their places and 9+1=10 is crossed off the list. We can do the same for 8+2. It is the same as 2+8. This continues until only five combinations remain. It is

sufficient to know these five combinations to know the combinations which form 10. The same is done for all numbers 2-18.

Control of error: Control Chart I. The child looks all the combinations he has made, noticing the tens in red on the diagonal.

STRIP BOARD (including doubles)

f. The Combinations of One Number with Zero as an Addend

Materials: addition strip board and strips
paper
Control Chart 1 and /or Chart II (45 combinations)

Exercise:
Let's find all of the combinations that make 9. 0+9=___? Our first addend is zero, so we place nothing on the board. Our second addend is 9, so we place the pink strip for 9 on the board. Our sum is 9. This continues until 9+0 is the last combination.

We notice that the first strip is all pink and the last strip is all blue.
Zero doesn't change the number in addition.
As before the child eliminates the unnecessary combinations.

Control of error: Control Chart 1 and/or Chart II

STRIP BOARD (including doubles)

g. Doubles of Numbers

Materials: addition strip board and strips
paper
Control Chart 1

Exercise:
We find the doubles of numbers by taking a blue strip: 1 and the same second addend; the pink strip for 1. The strips are placed on the board, and the combination is written on the paper. The sum is read at the top. The one strips are returned to their places, and the twos are added, etc.

N.B. Here the teacher helps the child reflect on his work, thus noticing that not only is 9+9=18, but also that 1/2 of 18 = 9. The possibility for dialogue here is very great and is a way of engaging langugae in the course of understanding mathematics.

Control of error: Control Chart II on which the double of each number is found at the top of each column.

GAMES AND EXPLORATION USING BEAD BARS

a. Complete List of Materials

Materials:
...Snake Game reminder beads, without the bridge
...box of ten bars
...box of colored bead bars (9 of each)
...box containing signs for the operations +, -. x, /, =, ()
...Addition Chart I

GAMES AND EXPLORATION USING BEAD BARS

b. The Snake Game

Materials:
...reminder beads usually black and white
...ten bars
...colored bead bars
...Addition Chart I

Presentation:
The reminder beads are lain out in a triangle arrangement to facilitate movement. The child is asked to make a snake with the colored bead bars. (The box is then covered again) The child no longer counts bead by bead to arrive at the ten. The first two beads are isolated. The child mentally computes the sum, i.e. 8+9=17. We can replace these with a ten (the ten bar is lain out) and seven (the reminder bead bar connects the ten to the rest of the snake). The 8 and 9 are put into the empty box. Go on adding the seven reminder bar to the next bar of the snake. These are isolated, added, the ten bar and corresponding reminder bar replace them in the snake, and the colored bead bar is returned to the empty box while the first reminder bead bar is replaced in its place in the triangle.
The child may make combinations of more than two bead bars, keeping the sum less than 19. If the child does not remember a combination, he may use the Addition Chart I. Command cards may be made.

Control of Error: As before, the child controls his work by matching. The ten bars and reminder bead bar (if any) are grouped together to one side. The colored bead bars from this snake are arranged in hierarchic order. the large box of colored bead bars is reopened, in case exchanges are necessary. The child takes a ten bar and a colored bead bar, i.e. 8. What must be added to eight to make ten? A colored bead bar of two is united with the 8 and placed next to the ten. If the child doesn't have a bead bar of two, an exchange must be made. Combinations of two numbers to make ten should always be used when controlling. The child sees that his addition was exact when the colored beads are all matched to ten bars (and the black reminder bar).

GAMES AND EXPLORATION USING BEAD BARS

c. Sums Less Than Ten

Materials:
...ten bars
...colored bead bars
...box of signs for operations

Presentation:
The teacher sets up an addition of two numbers, using colored bead bars and the plus and equals signs. the child reads the problem, computes the addition problem in his head and puts out the appropriate bead bar for the sum.

The child also adds combinations of these numbers. Sums are always less than ten. Command cards may be made for the work. The Addition Chart I may be used for control, or to help the child remember the combinations.

GAMES AND EXPLORATION USING BEAD BARS

d. Sums Greater Than Ten

Materials: same as above

Presentation:
As before, the child combines two or more numbers, whose sum will be greater than ten. The colored bead bar is placed below the ten bar to facilitate counting.
For example:

A child chooses 14 and 12 in beads to add. Tell the child to add the units first and then the tens to achieve the sum. 4 + 2 = 6 and 2 tens are twenty. The sum is 26.

The child chooses 18 + 25 in beads. Again units are added first. Since that sum is 13. Place a three bar below and carry the one ten mentally. Carrying the one ten add the other tens. The sum is 4 tens and the total sum is 43. This exercise gives a child experience with horizontal as well as vertical addition, a fact often overlooked in preparation for standardized tests.

Command cards may be made. The chart may be used as control.

GAMES AND EXPLORATION USING BEAD BARS

e. Changing the Order of the Addends

Materials: same as above

Presentation:
The teacher sets up an addition of two numbers, and the child completes the sum, placing the corresponding bead bars in their places. This equation is correct. The teacher switches the places of the two colored bead bars. Is it still correct? Perhaps it was a coincidence; let's try another.

Aim: to give the concept of the commutative property of addition (although it is not named as such at his age)

ADDITION CHARTS AND COMBINATION CARD EXERCISES

a. Passage from Chart I to Chart II

A group of children or individual children may copy Control Chart I or the teacher may make up 36 pink rectangles which are the dimensions of the space for the combinations.

Let's see how many combinations we can eliminate. We start from the first row 1+1=2. We must leave that...1+2=3. We can read along the diagonal the combinations that make 3. 2+1=3. We can cancel this combination. A card is placed over it, or it is crossed out. We'll go on to the combinations that make 4. They are 1+ 3 = 4, 2+ 2 = 4 and 3 +1 = 4. We must leave the first two. But what of 3 + 1 = 4 ? This can be eliminated.

We have another chart on which all of these combinations which were crossed out. are eliminated.

Something else is different. All of the combinations that make the same sum are arranged on a horizontal row. Now each column begins with a combination in which the addends are the same. But this chart contains all of the same combinations as before, just arranged slightly differently. Here you can find all of the combinations needed to do your work. The teacher gives an example or two to show that even though 8+2 is not listed, we find the sum when we look at 2+8.

ADDITION CHARTS AND COMBINATION CARD EXERCISES

b. Passage from Chart II to Chart III (the Whole Chart)

Materials:
...box of combination cards
...paper
...Control Chart I and III (which has only sums)

Exercise:
The child fishes for a combination. He reads it and writes it down on his paper. What is the first addend? Place your finger on the blue row at the first addend. Place your finger of the other hand on the pink column at the second addend. Move along the row and column until your fingers meet. The meeting place is at the sum. Write the sum on your paper. Fish again, etc.

Control of error: Control Chart I

ADDITION CHARTS AND COMBINATION CARD EXERCISES

c. Passage from Chart III to Chart IV (the Half Chart)

Materials:
...box of combination cards
...Control Chart I and IV (which has only sums)
...paper

Exercise:
In the same way as the child passed from Chart I to Chart II, a group activity may be conducted to show that some of these sums on Chart III can be eliminated. The combinations are reconstructed, going along the diagonal. This 4 was made by 1+3, this by 2+2, and this by 3+1, so we can eliminate it. When the elimination is complete, we have Chart IV.
The child fishes for a combination, reads it and writes it down on his paper, i.e. 5+9. On the pink column at the left, he places his finger on the first addend and goes all the way to the end of that row. Then he places his other finger on the second addend (in the same pink column) As before the two fingers are moved toward each other, and at the meeting place in the sum. When the child pulls out a combination, which has the first addend greater than the second, he will find that he can't do it. We simply reverse the order of the addends.

Control of error: Control Chart I

ADDITION CHARTS AND COMBINATION CARD EXERCISES

d. Passage from Chart IV to Chart V (the Simplified Tables)

Materials:
...box of combination cards

...Control Chart I and V (which has only 17 sums on two diagonals: external is even, internal is odd)
...paper

Exercise:
The child fishes for a combination, reads it and writes it down on his paper, i.e. 4+6. The first finger is placed at 4 on the left-hand column and moved along to the end of the row. The other finger is placed at the second addend, 6 on the left hand column, and is also moved along to the end of the row. The two fingers are then moved toward each other along the diagonal one step each at a time. Where they meet we read the sum. The next combination is 3+9=___ to emphasize that each finger goes the same distance. Then 2+9=___ is used as an example. The two fingers must meet on the internal diagonal. Other examples are given.

To bring a new slant of interest to this activity, the teacher brings into focus that when both addends are even, the sum is even. When both addends are odd, the sum is even. When one addend is even and the other odd, the sum is odd.

Control of error: Control Chart I

ADDITION CHARTS AND COMBINATION CARD EXERCISES

e. The Bingo Game for Addition (using Chart VI)

Materials:
...box of combination cards
...Chart VI (which has only the first and second addends; the rest is blank)
...Control Charts I and III
...box of 81 pink tiles for the sums

i. Exercise One:
The tiles are randomly arranged on the table face up. The child fishes for a combination, reads it and writes it down. The child thinks of the sum, looks for a tile with that sum , and looks for the place to put it on Chart VI. The first finger is put on the first addend, the other finger on the second addend. Where they meet is where the tile belongs. The sum is written on the paper. The child fishes again, etc.
ii. Exercise Two:
The tiles are in the box. The child fishes for a tile and reads the numeral. On his paper he writes the numeral and the equal sign. He thinks of a combination and writes it to complete the sentence. Then those two addends are used to find the tile's position.
iii. Exercise Three:
The tiles are placed in piles that have common sums. The child takes one pile, i.e. the pile of 8's. What does 8 equal? The child thinks of a combination, writes the sentence and uses the addends to find the corresponding position for the tile. He continues thinking of combinations until all of the tiles of that pile have been placed on the board. He notices that a diagonal is formed. the child does not need to do all the piles in one sitting; however he must complete whole piles he has chosen.
If the child arranges the piles in order, he may find an ascending and descending stair.

Control of error: Control Chart I and III

iv. Group Game One.
The teacher fishes for a combination, shows it to the child and asks, What is 2+3 equal to? If the child responds correctly, he receives the card (flash cards).
v. Group Game Two

The teacher fishes for a tile-say 10. What combinations are equal to 10? Each child gives a different combination until all have been named.

Age: Children's House-7 years

Aim: to give the child the possibility through many different exercises to memorize the combinations necessary for abstract problem solving

vi. Notes:
The pink strips on the addition strip board are segmented so that the child may see how many units are needed to make 10 and how many more are after 10. It is hoped that the child will absorb this aid to memorization. Later when the child is confronted with larger combinations, 24+8=?, he will have memorized 4+8=12 and the rest follows. The point of consciousness to be reached is to look for the combination which makes 10. Therefore, the child will say-I need to add 6 to 4 to make 10,
24 + 8 =
20 + (4 + 6) + 2
20 + (10) + 2

24+6 +(2) brings me up to 30. I have two more units on the right to add...32. Once the combinations are memorized, this type of mental activity naturally follows, thus abstraction. As the child works with various exercises, the teacher must observe and check to see if these points of consciousness are being met.

Bead Frame Addition

STATIC ADDITION

Materials:
...small bead frame, form

Presentation:
The teacher initially presents a static addition of two or three 4-digit numbers. The addition problem is written on the form. The first addend is formed on the bead frame. Now we must add the second number. Beginning with the units the teacher moves forward the corresponding number of beads, Invite the child to continue adding the tens, hundreds and thousands. The third addend is added in the same way, units first. The result is read and recorded appropriately on the form.

DYNAMIC ADDITION

A second (and all others succeeding) example is dynamic. The problem is written on the form as before and the first addend is formed on the frame. Of the second addend, begin by adding the units, counting the beads as they are moved forward...1,2,3,4 all of the units have been used, so we exchange a ten for ten more units- (move a ten forward, and ten units back) 5, 6, 7, 8, 9 (continue counting and adding unit beads) In fact, 6 + 9 = 15, which is 10 plus 5 units. Continue adding the second addend and then the third addend changing when necessary. Upon completion read and record the sum. The child should have many exercises of this type.

FINDING THE SUM MORE ABSTRACTLY

Presentation:
A dynamic addition problem is written on the form. This time add all of the units first, making the necessary changes. Write the total for the units under the units column and go on to add all of the tens. Where did these two tens come from? Those are the result of changes you made when adding up the units. (The carried over tens are not recorded anywhere.)
After each column the child records the total. The result when completely written corresponds to what is seen on the frame.

Game:
Form 999 on the bead frame. Add one unit. The result, 1000 was obtained by making three changes.

More Memorization Exercises

FURTHER EXPLORATION USING BEAD BARS

a. <u>Sums with Parentheses</u>

Materials:
...ten bars
...colored bead bars
...box of signs for operations

A. **Presentation**:
The teacher presents the new symbols: ( ) parentheses. First a combination of three numbers is set up and the sum is completed. These signs are called parentheses; they group things. The same addition is repeated, now with the first two addends in parentheses. Whenever you see these

parentheses in arithmetic, it means you must perform the addition inside the parentheses first. The first combination inside the parentheses is added, and the bead bar for the sum is placed below. The signs and bead bar for the next addend (outside the parentheses) are placed below as well. We find that this sum is the same as the original answer.
We haven't changed the addends, and the sum hasn't changed, only the addends have been grouped in a special way.

Aim: to give the concept of the associative property of addition

B. **Presentation**:
The teacher writes a long addition problem on a slip of paper. The child sets up the problem with the bead bars and signs, and computes the answer. The parentheses are placed around pairs of numbers. Review: We must perform the addition inside the parentheses first. New bead bars are put out for the sums of the pairs. These are added and the sum is found to be the same as the original problem

Note: Command cards may be made up for this work.

FURTHER EXPLORATION USING BEAD BARS

b. <u>Breaking Down the Addends</u>

Materials: same as before

Presentation:
On a strip the teacher writes a combination of two or three numbers. The child constructs the problem with bead bars and symbols and computes the answer and puts out the corresponding bead bars for the sum, The child reads the equation.
Now try to do this one; the teacher lays out combinations in parentheses that equal each addend of the first problem, i.e. 1st 9+7+8= 24, 2nd (4+5) + (3+4) + (6+2) = The child computes this as before, adding the first two addends in parentheses, placing the bead bar for the sum below, etc. Those three are added again to obtain the same answer as before. It is noticed that the first and third equations are identical. In the second, each addend was broken down into two smaller addends.

Aim: to give the concept of the dissociative property of addition

FURTHER EXPLORATION USING BEAD BARS

c. <u>Addends Greater than Ten</u>

Materials: same as before

Presentation:
The child is asked to set up a given addition with addends greater than ten. At first these should be static operations, i.e. 12 + 14 = When we have an addition like this, we first add the units, then the tens.
After practice such as this, we go on to dynamic operations, ex. 18 + 25 = First we add the units 8 + 5 = 13 We place a three bar here for the units and keep the ten in mind. We then add the tens; 1+2=3 and one ten in my mind makes four tens, put out four ten bars. The sum is 43.

Addition Chart: may be used here if the child doesn't remember a certain combination.

Aim: to practice carrying over
memorization of addition combinations

Notes: All of these games are more popular than the other memorization exercises because they are short, and the result is obtained quickly and easily. The children may invent others

Age: 6-61/2 years

SPECIAL CASES

Note: To find out if the child memorized not only the process of calculating in addition, but also the concept, the teacher organizes special combinations, starting with the combination that is familiar to the child.

Materials:
...addition combination booklet
...large sheet of paper or chart,
...red and black pens
...special combination cards

Presentation:
Taking any one page of the booklet, the teacher asks, 'When you work on this page, in this case 1+1, what are you looking for? the sum. In the work you have been doing up till now, you have been calculating the sum. On the chart the teacher writes the title, gives an example, and reads the example to the child.

0 - Calculating the Sum
1 + 2 = ? (One added to two gives you what number?)
The child fills in all the sums for that page.

1 - Calculating the Second Addend
1 + ? = 3 (One added to what number gives you three?)
Let's cover the column of second addends with a strip of paper. Note, this is the first time the child considers a problem of this type. What must we solve here? the second addend.

2 - Calculating the First Addend
? + 2 = 3 (what number added to 2 will give you 3?)
On the model page with totals, the column of first addends is covered. The child sees that in order to complete this combination, the first addend must be found.

3 - Inverse of Case Zero; Calculating the Sum
? = 1 + 2 (what number is obtained by adding one and two?)
The same column of combinations is written on another sheet, without their sums this time, and with the addends to the right of the equal sign. The child sees that the sum must be calculated, as in the first case. The difference is that the problem is set up in reverse order.

4 - Inverse of 1st Case, Calculating the Second Addend
3 = 1 + ? (3 is equal to one plus what number?)
The sums are written in this inverted model, and the column of second addends is covered. The child sees that he must find the second addend.

5 - Inverse of 2d Case, Calculating the First Addend
3 = ? + 2 (three is equal to what number plus 2?)
The column of first addends is covered.

6 - Calculating the First and Second Addends
3 = ? + ? (three was obtained by adding the first number to the second number. What were these numbers?)
In this last case, both columns of addends are covered with strips of paper, leaving only the sum.

Collective Activity:
The teacher passes out the combination cards. In turn the child reads the problem, states what must be found, and finds the case on the chart.

Individual Work:
When the child has understood all of the cases as presented, he may work with these special combination cards. He fishes for one, reads it and writes the equation in his notebook, substituting the red question mark for the right number written in red

Direct Aims: to memorize addition combinations
to make the child realize what must be calculated

Indirect Aim: to prepare first degree equations in algebra, i.e. (4 + x = 6).

Notes: On the chart, #0 is not a special case since this is what is familiar to the child. In cases #1, #2, #4, #5, and #6, subtraction is indirectly involved. For this reason fewer activities for memorization of subtraction are necessary.

Word Problems

In these problems the special cases previously examined are recalled. If the child has understood the last activity, he will be able to write a complete equation.

Examples:
1. Yesterday Adam had 3 notebooks. How many did his mother give him, if we know he has 5 notebooks today? 3 + ? = 5

Multiplication

Stamp Game Multiplication

Materials:
...wooden stamps of four types:
  ...green unit stamps printed with the numeral 1,
  ...blue tens stamps printed with the numeral 10,
  ...red hundred stamps printed with the numeral 100, and
  ...green thousand stamps printed with the numeral 1000
...box with three compartments each containing 9 skittles and one counter in the hierarchic colors;
...four small plates

Presentation:
Given a multiplication problem, the child prepares several like quantities, puts them together, makes the necessary changes and records the problem and the result in his notebook.

Memorization Exercises

MULTIPLICATION BOARDS (Bead Boards)

a. Introduction and List of Materials:

The child has encountered multiplication before. The first impression was given with the number rods, finding that the double of 5 is 10. Later with the decimal system material, the child learned that multiplication is a special type of addition. In the exercises that follow this concept will be reinforced and the child will be given the chance to memorize the necessary combinations.

Materials:
...Bead Board, and corresponding Green box which contains:
...100 green beads (the product), numeral cards of 1-10 (the multiplicand), and
one green counter (this, when placed by one of the xx...numerals 1-10 across the top, indicates the multiplier).
...Booklet of Combinations (10 pages of 10 combinations each)
...Box of Multiplication Combinations (same combinations as are found in the booklet)
...Box of green tiles for bingo game
...Multiplication Charts I-V (for control)

MULTIPLICATION BOARDS (Bead Boards)

b. Initial Presentation:

To familiarize the child with the materials, the teacher suggests a problem and writes it down, i.e. 3 x 4 = (three taken 4 times). The 3 numeral card is placed in the slot. The counter is placed over 1 as 3 beads are placed in the first column...(Attempt to get children at this point to be counting by threes up to whatever level they are capable, in place of counting every bead.) ...'three taken one time...' As the three beads are placed in each column, the counter is moved along, until '...three taken four times...' We've taken 3 four times, what is the product? The beads are counted and the result is recorded.

MULTIPLICATION BOARDS (Bead Boards)

c. Multiplication Booklets

Materials:
...Bead Board, and box with beads, cards, counter
...Combination booklet
...Chart I (for control; a summary of the combinations found in the booklet)

Exercise:
(Starting with 1 is a problem because it doesn't give the concept of multiplication) Start with any other unit like the number 3. The numeral card is placed in the slot. The child reads first combination 3 x 1 Three beads are placed in the first column with the counter over the numeral 1. The child records the answer in the booklet. The next combination is read: 3 x 2 =. The counter is moved over and three more beads are added. The product is recorded in the booklet. At 3 x 3 = the child should notice the geometrical form created when the multiplier and multiplicand are equal. If he doesn't,

the teacher may set up a situation wherein he may easily make the discovery himself forming several doubles in a row.

Control of error: Chart I

Note: In the material the child ends his work with the table of 10, rather than 9 as was the case in addition and subtraction. This is to show the simplicity of our decimal system. The table of 1 is very similar to the table of 10. It differs only in the presence of zero.

MULTIPLICATION BOARDS (Bead Boards)

d. Combination Cards

Materials:
...Bead Board and box with beads, cards, counter
...box of loose combinations
...Chart I (for control)

Exercise:
To facilitate the child's work in this exercise, the number cards used to indicate the multiplicand are arranged in a row or column on the table. The child fishes for a combination, 3 x 9 =, reads it, and writes it on his paper. The number card 3 is placed in the slot, the counter is placed over column 1; 3 beads are placed in the first column. The counter is moved to column 2, as three more beads are placed in that column, making a subtotal of 6. This continues up to 9. The result of 3 taken 9 times is 27. This product is written on the paper.

NOTE: In the beginning the teacher should supervise the child's work to see that he skip counts the beads as he goes along3, 6, 9, 12, 15, 18, 21, 24, 27. For if the child counts the beads one at a time when he is finished, he will never memorize the combinations.

The child removes the beads, number card and counter and fishes for another combination.

Control of error: When the child has finished his work, he controls with Chart I. This control reinforces memorization of the combinations.

SKIP COUNTING (Linear Counting)

a. List of Materials

Materials:
...Board of powers (though it is not named as such at this point)
...Cubes, long chains, squares, short chains
...Two Boxes:
    ...arrows for short chains, i.e. for 5 we have 1, 2, 3, 4, 5, 10, 15, 20, 25
    ...arrows for long chains, i.e. for 3 we have 1, 2, 3, 6, 9, 12, 15, 18, 21, 24, 27

b. **Initial Presentation**:
In the first presentation of the material the nomenclature should be given in a three period lesson, i.e. cube of seven, long chain of seven, short chain of seven, square of seven.

c. **Short Chain**:
The chain is lain stretched out on the table, and the child identifies it as, i.e. the short chain of 5. The teacher folds it up like a fan, and the child identifies it as a square of 5. This is proven by placing

a square on top of the folded chain to see that they are equal. The child is given the arrows to lay face up on the table. Together the teacher and the child put the arrows in their respective places as the beads are counted...1,2,3,4,5...; the counting continues by ones up to 10, and the arrow for 10 is placed there, and likewise counting by ones to 15. From 15, we add five more to reach 20, and place an arrow and add five more to 25, The square of 5 is placed at the end of the chain since the chain is equal to the square.

Activity:
Materials: same as before
The child lays out the arrows as before. Little by little he works from counting one by one using the arrows face up, to skip counting as he lays the arrows out, and then skip counting with all of the arrows face down.
When the child is able to skip count well with the arrows face down, he may also skip count regressively.

d. **Long Chain**:
The chain is lain out one the table or floor (if necessary), and the child identifies it as, i.e. the long chain of three. The child lays out the arrows appropriately as he skip counts. At 9, as the square of three is placed, the child is reminded if they do not see it, that this part of the chain makes a square. (It is equal to the short chain.) A square is placed by the chain at 9. The skip counting continues placing a square at 18 and finally at 27. The three squares are stacked up to see that they make a cube of 3, thus this chain is also equal to the cube of three. The cube is placed at the end of the chain.
In successive activities the child works up to counting progressively and regressively with the labels turned over.

e. **Games of Comparison**

Materials: same as before

Presentation:
The short chains are arranged as the pipes of the organ. Here the child sees the progression of the quantities which is the same as that seen on the shelves of the frame, in the hanging chains and in the cubes,
The visualization of the difference between the quantities becomes more apparent when the long chains are lain out in the same arrangement. In the long chains, the jump from one quantity to the next is more drastic.

Direct Aim: comparison of quantities sensorially

Indirect Aim: preparation for the powers of numbers (exponential increase)

BEAD BAR MULTIPLICATION

Materials:
...box containing colored bead bars 1-10, 55 of each
...Chart I (for control)

Presentation:
We are going to represent the table of a certain number with bead bars. The child is invited to choose a number, i.e., 8. We start with 8 taken one time. One 8 bar is lain horizontally 8 x 1 = ?8.

The product is also represented by an 8 bar (lain vertically below the first) The child writes 8 x 1 = 8.
Now take 8 two times. The two 8 bars are lain horizontally 8 x 2 = ?16. A ten bar and a six bar to represent the product are lain vertically, thus making a double row, The child writes the equation in his notebook. This continues until 8 x 10 = 80. Observe the geometric figures which have been formed with the 8 bars: 8x1 is a line; 8x2 is a rectangle; and so on; 8x8 represents a square, etc.

NOTE: Notice the rectangles that come before the square have a base longer than the height. The rectangles that come after the square have a base which is shorter than the height. 8x8 produced a square, which is when the number was multiplied by itself.

Afterward the child does the other tables.

Direct Aim: memorization of the multiplication tables
to bring the child to awareness of the functions of the multiplier and the multiplicand

Indirect Aims: to understand that a number when multiplied by ten results in the same number of tens and zero units
to realize that a number multiplied by itself results in a square to give the concept of forming surfaces, starting with a line, progressing to rectangles

BEAD BAR MULTIPLICATION

Adjunct: Multiplication by Ten

Materials:
...same box of colored bead bars 1-10
...Chart I
...small white strips of paper
...black pen

Presentation:
The child is invited to take 10 bars of any color he wants. Lay them in a column, skip counting as you go. When he is finished and has found the total, lay out the corresponding number of 10 bars vertically in a row beneath the column (note the resulting rectangles should be congruent)
The number we took was 4 (place a label for 4 beneath the row of beads), 10 times. The result was 40. How do we write 40? The 4 will now indicate tens, and we put a zero for the units. (place a 0 label next to the 4 to make 40)
Repeat this with other numbers until the child realizes by himself that when you multiply a number by 10, the product will be the multiplicand with a zero tacked on in the units place.

Direct Aims: memorization of multiplication
independent realization of the fact stated previously: n x 10 = n0

MULTIPLICATION CHARTS AND COMBINATION CARDS

a. Passage from Chart I to Chart II

The child copies Chart I. Later with the teacher or a group of children, they try to find those combinations which can be eliminated, that is, those which have like factors and equal products. Look at the first column 1 x 1 = 1 must remain. 1 x 2 = 2 and 2 x 1 = 2 are the same. 2 x 1 = 2 is crossed out. (Or the combinations to be eliminated may be covered with green strips of the appropriate size) As in addition we can change the order of the multiplier and multiplicand, eliminating many combinations. At the end we find that half of the chart is eliminated giving us Chart II. The combinations of two equal factors were not eliminated1 x 1 = 2, 2 x 2 = 4, 3 x 3 = 9This was the same case in addition. Chart II has only 55 combinations to be memorized. (We can make the child see that only 45 of these must be memorized, as the table of ten is simply a repetition of 1)

MULTIPLICATION CHARTS AND COMBINATION CARDS

b. Passage from Chart II to Chart III (the Whole Chart)

Materials:
...Chart III (products only- the numbers in pink serve as the multiplicand, blue as the multiplier; the one should be colored violet.   Along the diagonal are found the squares of the numbers)
...box of combination cards
...Chart I or II

Exercise:
The child fishes for a combination, writes it down on a piece of paper 5 x 7 =. A finger of the left hand is placed on the 5 (pink) while a finger of the right hand is placed on the 7 (blue). Where the fingers meet, the product is found. This is written on the paper to complete the equation. The child fishes for another combination, and so on.

Control of Error: Chart I or II

MULTIPLICATION CHARTS AND COMBINATION CARDS

c. Passage from Chart III to Chart IV (the Half Chart)

Materials:
...Chart IV (half of Chart III)
...box of combination cards
...Chart I or II

Exercise:
The child fishes for a combination and writes it down 8 x 3 =. I know that 8 x 3 gives me the same result as 3 x 8. One finger is placed on 3, another on 8 (both on the pink column) The top finger goes to the end of the row, then the two fingers come together. Where they meet, we find the product. This is written down. The child fishes for another combination, and so on.

Control of Error: Chart I or II

Note: At his point, to verify memorization, the child may be given command cards.
Example:

Find these products: 2x3= 5x6= 4x7=

etc.

and their inverses

MULTIPLICATION CHARTS AND COMBINATION CARDS

d. The Bingo Game of Multiplication (using Chart V)

Materials:
...Chart V
...box of tiles with products
...(a) box of combination cards
...Chart I and III

A. Exercise
All of the tiles are lain out on the table face up. The child fishes for a combination and writes it down 8 x 4 =. He thinks of the answer and writes it down. The tile with the product is found and placed on the board. The child fishes for another combination.

Control: Chart I for combinations, Chart II for placement.

B. Exercise
With all of the tiles in the box, the child fishes for a tile. On the paper he writes down a combination that will yield that product, 18 = 3 x 6. The tile is placed on the chart appropriately.

Control: Charts I and III.

C. Exercise
All of the tiles are placed in stacks of common products. The child chooses one stack, i.e. 12's. On the paper he writes down a combination that will yield 12, 12 = 2 x 6. One of the '12' tiles is placed on the chart where 2 and 6 meet. The child thinks of another, and continues until all of the tiles in the stack are used. Control: With Chart 1, we can check to see that all possible combinations have been considered.
Chart III controls placement.

Note: What shape is made when the stacks of tiles are lined up in order? No special figure is made this time.

Group Games:
As before the teacher may read a combination, the child responds with the correct product; or the teacher picks a tile. The children give all of the possible combinations. These games should be done frequently, as they encourage the child to go back to these exercise if he needs more practice.

Age: from 6-7 (this work lasts for one year)

MULTIPLICATION BY 10, 100, 1000

(Note: This activity is a prerequisite for the small bead frame)

Materials:
...decimal system materials
...paper
...black and red pencils

Presentation:
The teacher isolates a 10-bar. How many units are there in 10? 10. The teacher isolates a hundred square. How many tens are there in 100? How many units? Isolate the cube. How many hundreds are there in 1000? How many units? tens?
We can say that 10 tens is the same as 100, 100 tens is the same as 1000 and so on. With the child draw relative conclusions of all the changes possible.

Aim: to be sure that the child has understood the concept of change

By ten
Write down a multiplication problem and ask the child to lay out the problem, using the golden bead material i.e. (21 x 10 =). The child, knowing the function of multiplication, combines these quantities and makes the necessary changes. With the answer - two hundreds, one ten, and the zero is written in red. 21x10 = 21**0** Observe that the product is simply 21 (the multiplicand) with a zero after it.
Do many examples of this type, including: 30 x 10 = 300

By one hundred
Write down a multiplication problem such as 23 x 100 = and ask the child to lay out the material.. We can't put out 23 one hundred times, we would run out of beads! We can multiply each unit by 100. Isolate one bead from the 23.
1 x 100 = 100 Substitute the bead for a hundred square. Repeat for the other two units. Then 10 x 100 = 1000. Replace each ten bar with a thousand cube, and so on. Record the product. 23 x 100 = 23**00**. Notice that the product has the same number of zeros as the multiplier.

By one thousand
Write the problem 4 x 1000 =. As before, multiply each unit by 1000, replacing each bead with a thousand cube. Record the product 4 x 1000 = 4000. In this case we jumped from the units, past the tens, past the hundreds, to the thousands. For each hierarchy that we increased, one zero was added. Observe as before that the number of zeros in the product is the same as the number of zeros in the multiplier. The product is simply the number of zeros in the multiplier.

Direct Aim: ease of multiplying by powers of ten, and understanding of the characteristic patterns of such multiplication.

Indirect Aim: preparation for multiplication using the bead frames

Checkerboard - Geometrical Analysis of Multiplication

INTRODUCTION TO THE CHECKERBOARD

Materials: The Checkerboard:
each square has sides of 7 cm so that the longest bar will fit;
squares of the same color on the diagonal represent the same value;
the numbers along the side and the bottom are printed in hierarchic colors the bottom right square, which is green and represents simple units is the square having the least value;
whereas the square in the opposite corner, which is also green and represents units of billions, is the square having the greatest value

also, box of bead bars, 55 of each (remove the 10-bars)
box containing numeral cards 1-9
3 series printed in black on white cards (multiplicand)
3 series printed in black on gray cards (multiplier)

Presentation:
Familiarize the children with the board, noting the value of each square, the hierarchic colors, and the pattern of the values along the diagonal. Afterward the child may draw his own board.

Games:
A) Place a bead bar such as (5) on the unit square and ask the child to identify its value. Move the bead bar to the left to the tens square. Identify its value. (50) Move the bead bar along the diagonal to the next tens square. Identify its value. (50) Place the bead bar on the bottom row-hundred square. Continue moving the bead bar, and identifying its value as it changes its position.

B) Place a bead bar on the unit square and identify its value. As it moves up the column, identify its value. Note that the value increases by 10 each time. Repeat the procedure moving the bead bar down the column, noting that the value decreases by 10 each time. Move the bead bar to the ten square at the bottom and repeat the game. Again we notice that the value increases by 10 as it goes toward the top, and it decreases by 10 as it moves toward the bottom again.

C) Place two bead bars on two different squares and read its value. Place two bead bars in such a way that an inferior hierarchy is left blank.-430,403.

D) Place four bead bars on four different squares along the bottom row. Identify the number. Move one bar to the second row and identify the value; it is the same. Continue moving one bead bar at a time along the diagonal, identifying the number; it stays the same.

Aim: to familiarize the child with the board
to emphasize that squares on the diagonal have the same value

Note: With the bead frames and the hierarchic materials (blocks) we gave the concept of the hierarchies. With this material we will reinforce that concept. Since the concept is presented in a different way, we must be sure that the child understands how this work is organized.

MULTIPLICATION WITH THE CHECKERBOARD

Materials:
...checkerboard
...box of numeral cards 1-9, gray and white
...box of bead bars 1-9, 55 of each

Presentation:
Propose a problem: 4357 x 23 =

a. 1st level
Form the multiplicand by using the white cards placed on the appropriate numerals
on the bottom edge of the board. (7 is placed on *1,* 5 is placed on *10,* etc.) Form the multiplier using the gray cards placed on the appropriate numerals on the right edge of the board.
Begin multiplying with the units. First we take 7 three times. Place 3 seven bars on the unit square.
5 x 3place 3 five bars on the tens square
3 x 3place 3 three bars on the hundreds square
4 x 3place 3 four bars on the thousands square

Keep a finger on the digit of the multiplicand to remember your place. Notice that there are three of each quantity in this row. Why? because the multiplier is 3. Since we have finished multiplying by the units, we can turn over the gray card. Continue multiplying by the tens noting the value of each square (this emphasis is important): tens multiplied by units give tens, tens multiplied by tens gives hundreds, etc. Notice that 2 dominates the row. Turn over the card.
Move the bead bars of the upper row along the diagonal to the bottom row. Beginning with units make changes to total the product, carrying over as necessary, i.e. the bead bars in the ten square total 3131 tens. How do we express 31 tens in conventional language? Three hundred ten. So, place a unit bead in the ten square, and a 3 bar in the hundred square.
Read the total and record the product.

Aim: to understand the process of multiplication using the board.

b. 2nd level-Small Multiplication
Propose a problem: 4357 x 423 = and set up the board with the numeral cards.
Begin multiplying with the units, but this time only put out the bead bars for the product.
7 x 3 = 21 put a unit bead in the unit square, 2-bar in tens
5 x 3 = 15 5-bar in tens square, unit in hundreds
3 x 3 = 9 9-bar in the hundred square
4 x 3 = 12 2-bar in thousands, unit in ten thousands
Turn over the gray card. Continue with the tens. Move the bead bars along the diagonal in the end.
Make the necessary changes and read the final product.

c. 3rd level-Partial Products (this passage can be skipped)
Multiply in the same way as before (2nd level). After everything in the multiplicand has been multiplied by the units, make the necessary changes in that row and record the partial product, Continue with the tens, etc. After all of the partial products have been recorded, move the bead bars along the diagonal to the bottom row.
Make the changes and read the total product.

d. 4th level-Mental Carrying Over
The procedure is different from the 3rd level only in that the child carries mentally. 7 x 3 = 21 put the unit bead down, remember 2...5 x 3 = 15plus 2 = 17. etc. The partial product is read without making any changes.

MULTIPLICATION AND DRAWING

Materials:
...checkerboard
...box of bead bars 1-9, 55 of each
...box of numeral cards 1-9, gray and white
...graph paper
...colored pencils in red, green, and blue

Presentation:
The multiplication is done in the same manner as the first level of checkerboard multiplication. Propose a problem and write it down. Set up the numeral cards on the board and begin multiplying by the units of the multiplier (place 3 bars of 2 for 3 x 2, etc.) Draw the result of this partial product - 1 square for each bead. Color the rectangles and squares in the appropriate hierarchic colors: units multiplied by units gives units, so color it green, etc. Write each product in the rectangles.
Analyze the first partial product: units times units gives us units (write 1 x 1 = 1); 2 x 3 = 6; these are 6 units tens. Tens times units gives us tens (10 x 1 = 10); 3 x 3 = 9; 9 tens = 90. Continue analyzing the partial product in this way.

Go on to multiply by the tens, placing the bars on the checkerboard. Draw the result and write the products of the small multiplications. Analyze the partial product in the same way as before.
Make the necessary changes in the first row to obtain the partial product. Verify this by adding the column of products (in the analysis of the first partial product) Write this partial product under the multiplication problem. Repeat the procedure for the second partial product.
Move the bars along the diagonal to the bottom row. Make the changes to get the total. Add the two partial products to control and verify. Record this final product under the original problem.

3432	u x u = u	1 x 1 =1	2 x 3 = 6	6
x 43	t x u = t	10 x 1 =10	3 x 3 = 9	90
10296	h x u = h	100 x 1 =100	4 x 3 = 12	1200
137280	th x u = th	1000 x 1=1000	3 x 3 = 9	+ 9000
147576				10296
	u x t = t	1 x 10 =10	2 x 4 = 8	80
	t x t = h	10 x 10 =100	3 x 4 = 12	1200
	h x t = th	100 x 10=1000	4 x 4 = 16	16000
	th x t = tth	1000 x 10=10,000	3 x 4 = 12	+120000
				+137280
				147,576

Having completed and understood this activity, the child should have realized what multiplication must be done to change from one hierarchy to another: to obtain hundreds, he has three possibilities as indicated on the checkerboard: 100 x 1, 10 x 10, 1 x 100.
He also should realize that the quantities are moved along the diagonal to add quantities of the same hierarchy. This is a change from adding in vertical columns on the forms for the bead frame. The colors, however, aid the understanding of this difference.

This drawing activity allows the child to visualize all multiplication geometrically as rectangles and squares. Even the square of a number with 2 or more digits is composed of smaller squares and rectangles. Thus this work is remotely indirect preparation for square roots and the study of perfect squares.

Age: 7-8 years [For all checkerboard work]

Aim: to reinforce the concept of hierarchies
to visualize multiplication in its geometric form

Bead Frame Multiplication

SMALL BEAD FRAME

a. **Multiplication By 10, 100, 1000**

Materials:
...small bead frame
...paper and pencil

Presentation:
Write down the problem 2 x 10 =. Perform this on the frame by sliding forward groups of two beads, changing as necessary. Record the product 2 x 10 = 20. Repeat the process in the problem 20 x 10 =. Record the product 20 x 10 = 200. Try this problem
2 x 100 =. It would take all day and most of the night to bring forward 100 groups of two beads. Recall from the multiplication game-Multiplication by 10, 10, 1000 the simple way to do this. Slide

forward the 2 beads to correspond to the multiplicand. We can multiply 1 x 100 and get 100-slide the one unit back, and slide one hundred forward. Repeat for the second unit. Record the product 2 x 100 = 200. We went from units to the hundreds. How many jumps did we have to take? 2 How many zero's are in the product? 2 The two zero's indicate that we have passed two hierarchies.
Do many other examples, i.e. 2 x 1000 = to be sure that the child is very comfortable and familiar with this process.

Note: We are limited by the frame to having a multiplicand of only one digit when the multiplier is 1000 and vice versa.

Aims: to understand of the use of the small bead frame in performing multiplication
to reinforce the concepts of the relative positions of the hierarchies, and changing from one hierarchy to another

SMALL BEAD FRAME

b. Multiplication with a One-Digit Multiplier

Materials:
...bead frame
...small form

Presentation: To isolate the difficulty of decomposing the multiplicand, we begin with a static multiplication. From then on the child will work with dynamic problems.

2321 x 3	Write the problem on the left side of the form.

The first thing we must do is to decompose the multiplicand. There are how many units? 1, we write 1 on the right side under units. All of this we must multiply by 3. On the bead frame, perform the multiplication. 1 x 3 = 3, move forward three units beads.
2 x 3=6, but 6 what? 6 tens! Move forward 6 ten beads, etc. (By this time the child should have memorized the combinations and should bring forward the product of the small multiplication) Read the product and record it on the left side of the form.

Try a dynamic multiplication

2463 x 4	Decompose the multiplicand in the same way as before.

Perform the multiplication 3 x 4 = 12, 12 is 2 units and 1 ten...6 x 4 = 24, 24 what? 24 tens4 tens and 2 hundreds, etc.
Read the product on the frame and record it.

Experiment:
Try performing any one of these multiplications out of order, i.e. 6 x 4 = 24 tens,
2 x 4 = 8 thousands, 3 x 4 = 12 units and 4 x 4 = 16 hundreds. The product is still the same.

Note: Maria Montessori said, "When you go to the theater, you find that people are all sitting in different areas; some are in the balcony, some are in the boxes. Why? Each person has chosen a seat by buying a certain type of ticket. In the same way, these units must be in the top row of the bead frame. That is their fixed place."

Age: 6-7 years

Aim: realization of the importance of the position of each digit

LARGE BEAD FRAME - MULTIPLIERS OF 2 OR MORE DIGITS

a. The Whole Product

Materials:
...large bead frame
...corresponding long form
...red and black pencils

Presentation:
Write a multiplication problem on the form

8457 x 34	Decompose the multiplicand in the same way as before.

On the right side decompose the multiplicand as before. First decompose the number for multiplication by 4 units.

7 30 400 8000	x 4

We must also multiply the multiplicand by 30; decompose the number a second time below the first. We know that we cannot multiply by such a large number on the bead frame. The rule is that we must always multiply by units. 7 x 30 is the same as 70 x 3. (7 x 30 = 7 x 3 x 10 = (commutative property) 7 x 10 x 3 = 70 x 3) So we can write this decomposition in a different way. For our work we will use the first and third decompositions.
Note: By decomposing the multiplicand we have reduced the problem to a series of small calculations at the level of memorization.

Begin multiplying

7 x 4 = 28 28 units move forward 8 units, 2 tens
3 x 4 = 12 12 tens. move forward 2 tens and 1 hundred
4 x 4 = 16 16 hundreds. move forward 6 hundreds and 1 thousand
8 x 4 = 32 32 thousands move forward 2 thousands and 3 ten thousands

Continue with the second decomposition in the same way.

Read the final product and record it.

Note : This multiplication can be shown on an adding machine in the same way, though as a repeated addition. Calculators operate on the same principle of moving the multiplicand to the left and adding zeros.

The child may go on to do multiplication with multipliers of 3 or more digits as well. With a three-digit multiplier there will be 5 decompositions of which only the 1st, 3rd, 5th will be used for the multiplication on the frame.

LARGE BEAD FRAME - MULTIPLIERS OF 2 OR MORE DIGITS

b. Partial Products

Presentation: The child by this time should have reached a level of abstraction with column addition.
The procedure is exactly like the first, except that the child will stop after each multiplier and record the partial product, clearing the frame he begins multiplying with the next multiplier. When the child records all of the partial products, he adds them to find the total product.

4387
x 245
21935
175480
877400
1074815

Here we can observe that the first partial product which was the result of multiplying the units has its first digit under the units column. The first digit (other than zero) of the second partial (which was the result of multiplying by the tens) is under the tens column, etc.

Age: 7-8 years, or when the child is adding abstractly

HORIZONTAL GOLDEN BEAD FRAME

a. The Whole Product

Materials:
...The Horizontal Bead Frame, which lies flat on the table.
It is less sensorial in that hierarchic colors and spaces between the classes have been eliminated.
...box of 4 series of gray cards on which 1-9 is written in black (to serve as multiplier)
...strips of white paper on which multiplicand will be written.

(note: the black lines are drawn on the board beneath the wires; they will indicate where to begin the multiplication when multiplying by units, tens, hundreds or thousands.)

Presentation:
All of the previous operations can be done with this material, but we will do the most interesting: multiplication with a two-digit multiplier.

Write down a problem, 6542 x 36 = and show the child how to set up this problem. Place a white strip over the zeros and secure it with a rubber band or tape. Write the multiplicand on the strip so that the digits correspond to the correct wires. Find among the gray cards the digits needed to form the multiplier. Place the 6 over the lowest green dot which represents the units, and the 3 over the blue dot for the tens. The beads should be at the top to start.
Begin multiplying 2 x 6 = 12, bring down 2 units and 1 ten. 4 x 6 = 24 tens - bring down 4 tens and 2 hundreds and so on. After the multiplicand has been multiplied by the units we can turn over the card '6'.
In order to multiply by the 3 tens, 30, we must move the multiplicand to the left one space to let one red zero show. This is just like multiplying the number by 10.
The black line indicates that we start with the row of tens. Continue multiplying, making changes as necessary. In the end we read the product and record it.

HORIZONTAL GOLDEN BEAD FRAME

b. Partial Products

Materials:
...The Horizontal Bead Frame
...box of 4 series of gray cards on which 1-9 is written in black
...strips of white paper

The procedure followed here is exactly the same, except that when the child has finished with one multiplier he turns over the card, reads the partial product, writes it and clears the frame before beginning with the next multiplier. In the end he adds abstractly to total the partial products.

Age: 7-8 years

HORIZONTAL GOLDEN BEAD FRAME

c. Carrying Mentally

Materials:
...The Horizontal Bead Frame
...box of 4 series of gray cards on which 1-9 is written in black
...strips of white paper

The child sets up the multiplication problem on the frame.

2443
x 46

3 x 6 = 18 move down 8 units, remember one ten in your head.

5 x 6 = 30...+1 = 31 move down 1 ten, remember 3 hundreds

4 x 6 = 24...+3 = 27 hundreds-move down 7 hundreds, etc.

Record the partial product and clear the frame before beginning multiplication by the tens.

Age: 8 years

Note: The work done with this frame is on a higher level of abstraction than the work with the hierarchic frames. In both activities the tens, hundreds and thousands of the multiplier were reduced by a power of 10, while the multiplicand increased by a power of 10. The same work was done in two different ways.
At the end of this work the child should understand that when he starts multiplying with a new digit of the multiplier, he must move over one hierarchy. The partial products must start from the same hierarchy as the corresponding digit of the multiplier.
This activity forms the basis for an understanding of the function of multiplication with a multiplier of two or more digits, and a preparation for abstract solution. The child doing this activity will be stimulated to invent his own problems.

More Memorization Exercises

THE SNAKE GAME

Materials: same as for addition snake

Exercise:
This time in constructing the snake, it is not important to use many different colors; rather several bead bars of each of a few colors should be used. A resulting snake may be: 2 + 4 + 8 + 4 + 2 + 4 +8 + 4 + 2. The snake is counted as in addition.

Control of Error: To one side the ten bars and black and white bars are grouped together. At the other side the original bead bars are grouped according to color. How many times do we have 8? We can say 8x2; the equation is written on a piece of paper. The beads to represent the product 16 are placed below the group of 8 bars. The same is done for 4x4 and 2x3. The three products are added on paper and/or with the second row of bead bars to show that the snake was counted correctly.

Age: from 6 years onwards

Note: The child does this work after completion of the exercises for memorization of multiplication

VARIOUS WAYS OF CONSTRUCTING A PRODUCT

Materials:
...box of colored bead bars 1-10
...Chart I

Presentation:
The teacher writes a product on a piece of paper and places it on the table (it will be the heading). Let's see how many ways we can make 12? Shall we try with one? The child should see that any number can be made using one, therefore it will be omitted for this game.
Beginning with two, the child tries to make 12 by repeating 2, skip counting as he takes out the bars. Stopping at 12, he finds that 2 taken six times equal 12. These bars are placed in a column under the strip. the child writes the combination in his notebook,
12 = 2 x 6, which is read 12 is constructed by taking 2, 6 times.
Go on trying with 3 and 4 following the same steps. When the child tries with 5, he will find that 15 is greater than 12, so the bead bars are put back in the box. Go on with 6. Trying with 7, the child

finds that it won't work, because at the second bar we've already exceeded 12. Thus we must stop at 6.

Control of error: The child looks on Chart I, finds the combinations he has made, and the absence of 12 in the 5 column, 7 column and so on.

Direct Aim: memorization of multiplication

Indirect Aims:
...preparation for memorization of division
...preparation for decomposition into prime factors
...preparation for the study of multiples

SMALL MULTIPLICATION

Materials:
...box of colored bead bars 1-10
...Chart I

Presentation:
The child is invited to take any number (not to exceed 10) of any color of bead bars. From this pile the child begins to lay them out in a column; saying 4 taken 1 time, putting out a bar and writing either the equation or just the product in his notebook, He continues until the whole pile has been lain out.

Control of error: Chart I. If the child wrote the equation then he has written a table. If only the products were recorded, then he has done progressive numeration (skip counting).

Direct Aim: memorization of multiplication

Indirect Aims:
...preparation for memorization of division
...preparation for decomposition into prime factors
...preparation for the study of multiples

INVERSE PRODUCTS

Materials: same as above

Presentation:
The child is asked to take the bar of 7, 6 times. The bars are lain in a column. How do we express this? 7 x 6 = 42 is written down. Now take the bar of 6, 7 times. These are lain next to the others. When the child writes down 6 x 7 =42, he should observe that the product was the same. Are the multiplication problems the same? Notice that the two rectangles are equal, each having 42. even though the order of the factors is not the same. The child should observe in the written form that the factors are the same, just reversed in their positions.
Give several examples of this.

Direct Aims: memorization of multiplication understanding of the commutative property of multiplication

CONTRUCTION OF A SQUARE

Materials:
...box of colored bead bars 1-10
...one each of the squares 1-10

Presentation:
Let's try to multiply all of the numbers by themselves. During this activity the child writes the equations as he goes along. He may also draw this on graph paper as he progresses or when he is finished.
Start with one. One taken one time is one. Put out one bead. Write down 1 x 1 = 1. Two taken two times is four. Place two 2- bars in a column next to one. Write down
2 x 2 = 4. Continue in this manner until 10 x 10, resulting in 9 columns in a row. If the child doesn't remember a combination he may check Chart I.
We have multiplied all the numbers by themselves. What have we formed? squares. Because these are only bars, we can substitute them with the real squares. With one, there is no square because 1 x 1 is just 1. Replace the 2 bars of 2 with the 2 square and so one up to 10.

Direct Aim:
...memorization of multiplication
...realization that a number taken by itself n = makes a square.

Indirect Aim: preparation for the powers of numbers

MULTIPLICATION OF A SUM

Materials:
...box of colored bead bars 1-10
...box of signs for the operations
...Chart I
...pieces of white paper
...pen

Binomial Presentation:
The teacher writes on a strip (5 + 2) x 3 =, which is read, take 5 plus 2, 3 times. This is a multiplication problem. The problem is prepared as before with bead bars for the addends, signs, parentheses and the multiplier, written on a little card. Recall that the operation inside the parentheses is done first. The child places the 7-bar for the sum under the parentheses, places signs, multiplier and the product, represented in bead bars. The work is recorded.

On a different day:
When you find a problem of this kind, you can also multiply one term at a time by the multiplier. The other way will be put aside for now. (The equation in beads and cards:
7 x 3 = 21 is placed off to the side, leaving the slip of paper and the original layout of beads)
First take 5, 3 times, 5 x 3 is written on a strip and 3 bars of 5 are placed below the original 5 barplus (put out the sign)2 taken 3 times, 2 x 3 is also written on a strip and 3 bars of 2 are lain out. Now we must find these products. The products are placed below the group in a perpendicular position. Add 15 + 6 and put out the result, The result is the same as the equation we put aside. The child writes in his notebook:

(5 + 2) x 3 =
(5 x 3) + (2 x 3)

	=
15 + 6 =	

Trinomial Presentation: On yet another day
The teacher writes a problem on a strip such as: (5 + 2 + 3) x 4 =. The child lays out the corresponding beads for 5, 2, and 3, the signs, parentheses and a little card for 4. As before we must multiply each term by the multiplier. Then, for control, the child may add the addends within parentheses and multiply the sum by 4. When his work is written in his book it should be:

(5 + 2 + 3) x 4 =	(__+__+__) x 4 =
(5 x 4) + (2 x 4) + (3 x 4) =	beads
20 + 8 + 12 =	40 beads

Note: After the child has learned to multiply such a problem term by term, he should not go back to the first way of adding first, then multiplying. In this way the following aims will be achieved.

Direct Aim: memorization of multiplication
understanding of the distributive property of multiplication over addition

Indirect Aim: preparation for the square of the polynomial

ANALYSIS OF THE SQUARES - BINOMIAL

Materials:
...box of colored bead bars 1-10
...one each of the squares of 1-10
...rubber bands

Presentation:
The teacher presents the hundred square, and the child identifies it as the square of 10, or the hundred square. Observe that it has 10 beads on one side and ten on the other. Write 10 x 10 = 100 on a strip of paper.
Because we know this square so well, we are going to perform a small division of the square. The child is asked to count 6 beads along one side. A rubber band is placed after the 6th bead around the square, (Note: the result will be two perpendicular rubber bands.)
How many parts has the square been divided into? Let's see how the 4 parts are composed: 6 on one side, 6 on the other, 6x6; 4 by 6 or 4 taken 6 times , 6x4, 4x4. Let's write this down and while writing, we can reconstruct the square with colored bead bars, As each combination is written, the product is recorded, as the appropriate beads are lain out.

6 X 6 = 36	4 X 6 = 24
6 X 4 = 24	4 X 4 = 16

60 + 40= 100

Add the products. Push the bead bars together and place the 100 square over it to verify sensorially that the decomposition was done correctly.

Note: Later, after the passing of a year and much work with the decomposition of a square and the powers of numbers, the child will learn the exact way of writing this:

$$10^2 = (6 + 4)^2 = (6 + 4) \times (6 + 4) = (6 \times 6) + (6 \times 4) + (4 \times 6) + (4 \times 4)$$

ANALYSIS OF THE SQUARES - TRINOMIAL

Materials:
...box of colored bead bars 1-10
...one each of the squares of 1-10
...rubber bands

Presentation:
This time we'll decompose the square in a different way. Count 3 along one side, and place the rubber band around the square. Count 3 along the other side and put on a rubber band. Now continue counting 5 more beads, and put a rubber band around. Do the same on the other side. Into how many parts have we decomposed the square? Observe how each of the 9 parts is composed. Notice the three squares which all lie on the diagonal, and the various rectangles formed. As before, we'll construct the squares as we write it all down.

3 x 3 = 9	5 x 3 = 15	2 x 3 = 6	
3 x 5 = 15	5 x 5 = 25	2 x 5 = 10	
3 x 2 = 6	5 x 2 = 10	2 x 2 = 4	
30	+ 50	+ 20	= 100

Control of Error: Compute the sums of the three columns and add them together. Then slide all the bead bars toward the center, and place the 100-square on top. These are the two ways to prove that this equals one hundred.

PASSAGE FROM ONE SQUARE TO A SUCCEEDING SQUARE

Materials:
...box of colored bead bars 1-10
...one each of the squares of 1-10
...rubber bands

Presentation:
The child identifies the square chosen by the teacher, i.e. the square of 4. Let's use this square to build a square of 5. Allow the child to suggest and discover ways of doing this. Guide the work, giving guidelines such as: 'The sides must be built upon.'
Four bars are placed to the right and the bottom of the square. A one-bead fills in the hole left in the lower right hand corner. Superimpose the square of five to verify successful completion. What was added to the square of 4 to make the square of 5? two bars of 4 and one unit.

square of 4 + what was added = square of 5	4 x 4 = 16 4 + 4 + 1 = 9 5 x 5 = 25

Aim: indirect preparation for the square of a binomial

PASSAGE FROM ONE SQUARE TO A NON-SUCCESSIVE SQUARE

Materials:
...box of colored bead bars 1-10
...one each of the squares of 1-10
...rubber bands

Presentation:
The teacher volunteers to assist in constructing a square of 9 from a square of 5. What is the difference when you take 5 from 9? So, how many more bars of 5 must be added to this side to make 9? Four bars of 5 are added to the right side. In the same way 4 5-bars are added to the bottom. Again, there is a hole. Count the number of beads on the sides of the hole...4 by 4. Four 4-bars can be replaced by the square, Use the square of 9 to control:

	5 x 5	+ 5 x 4	+ 5 x 4	+ 4 x 4
What was added?	25 +	20 +	20 +	+ 16 = 81
		9 x 9 = 81		

SKIP COUNT CHAINS - FURTHER EXPLORATION

a. <u>List of Materials</u>

Materials:
...Board of powers (though it is not named as such at this point)
...Cubes, long chains, squares, short chains
...Two Boxes:
  ...arrows for short chains, i.e. for 5 we have 1, 2, 3, 4, 5, 10, 15, 20, 25
  ...arrows for long chains, i.e. for 3 we have 1, 2, 3, 6, 9, 12, 15, 18,21, 24, 27

SKIP COUNT CHAINS - FURTHER EXPLORATION

b. <u>Construction of Geometric Figures</u>

Materials: short chains

Exercise:
Using the short chains the child tries to construct regular polygons. With the short chain of 2, he cannot make anything, though with 3, he can make a triangle; with 4, a square; with 5, a pentagon, and so on.

Direct Aim: reinforcement of law: the smallest possible polygon must have three sides

Indirect Aim: preparation for perimeters of polygons: preparation for multiples and divisibility

SKIP COUNT CHAINS - FURTHER EXPLORATION

c. Decanomial (a polynomial having ten terms) & The Construction of Chart I

List of Materials

Note: This material is presented parallel to memorization of multiplication, in three different presentations.

Materials:
...box of bead bars, 1-10, 55 of each
...square of the numbers 2-10
...Multiplication (memorization) Charts I and III

SKIP COUNT CHAINS - FURTHER EXPLORATION

c. Decanomial (a polynomial having ten terms) & The Construction of Chart I

Vertical Presentation

Beginning with the ones table, reconstruct Multiplication Chart I, 1 x 1 =...2 The child states the product and puts out one bead, 1 x 2 = ...2 The child takes a unit bead (1) two times and places them in a column, 1 x 3 = ...3 The child puts out three unit beads in a column and so on to 1 x 10 = 10.
Construct the twos table 2 x 1 =...2 Place the two bar at the beginning of a new column, so that it lines up with 1 x 1. Go on making a column for the table of 2...2 x 2, 2 x 3, 2 x 4, 2 x 5, 2 x 6...making new columns for each multiplicand, ending the last column with 10 x 10. Use Chart I for control, as you go along, if the child needs it for recalling products. The result will be a square, made up of 10 vertical strips each strip being the width and color of that bead bar.

SKIP COUNT CHAINS - FURTHER EXPLORATION

c. Decanomial (a polynomial having ten terms) & The Construction of Chart I

Horizontal Presentation

This time, the multiplier will remain constant, as we progress along the rows of Chart I. Begin with 1 x 1=...1 Place the unit bead 2 x 1 = 2 Place the two bar next to it forming a row. Go on to 10 x 1 =...10. Beginning the second row with 1 x 2, each of the bars must be taken twice. Place the unit beads in a column forming the beginning of the second row. Continue in this way up to the end of the last row-10 x 10
The result of this work is the same square as before. We can't tell by looking at the finished product whether it was made the first way or the second way. This square was formed by 10 horizontal strips, varying in width as before, but now with each having the same multi-colored pattern.

SKIP COUNT CHAINS - FURTHER EXPLORATION

c. Decanomial (a polynomial having ten terms) & The Construction of Chart I

Angular Presentation

As always we begin with 1 x 1 = 1. Go on to 2 x 2 = 2; put out a two bar, making a row, 2 x 1 gives us the same as 1 x 2; put out two unit beads making a column 2 x 2 =...4; put out two 2-bars. Outline the formation with a finger to help the child to observe the square that was formed. Continue with 3 x 1, then 1 x 3...3 x 2, then 2 x 3 and finally 3 x 3 to fill in the arrangement to make a larger square. Continue in this way with the child stating the products for each combination and stopping to observe each square that is formed.
The result of this work is the same square as before. Notice the various geometric forms; point, lines, rectangles, and squares. Substitute real squares for the bars: 1 x 1 is still 1 so we leave the bead. Replace 2 x 2, 3 x 3 and so on up to 10 x 10. We can see that the squares are placed on the diagonal just as they are on Chart III. (outlined in bold black lines)

Control of Error: visible arrangement; number of bead bars in the box

Direct Aim:
...memorization of multiplication
...development of mental flexibility

Indirect Aim: preparation for the Decanomial

SKIP COUNT CHAINS - FURTHER EXPLORATION

c. Decanomial (a polynomial having ten terms) & The Construction of Chart I

Commuted Decanomial

Materials:
...Two boxes of bead bars 1-10, 55 of each
(actually 1 box of 55 each, and nine 8-bars, twenty six 9-bars, and forty five 10-bars)
...squares and cubes of the numbers 2-10
...Multiplication Charts I and III

Part One
Presentation:
The Decanomial is lain out already with the squares substituted along the diagonal.
"We're going to change its beautiful colors. As usual we start with 1x1. It stays the same. Here we have 2x1 which is a green bar. 1x2, which is the same thing as 2x1, is a poor imitation of a bar. We'll exchange this fake bar for a real 2-bar. Then we have 2x2, which is a square; it remains the same."
Continue with 3 x 1, and 1 x 3 which are the same thing. Therefore, we'll make them look the same by placing a pink 3-bar in place of these unit beads. We have 3 x 2 and 2 x 3 which are both equal to 6, because changing the order of the factor doesn't affect the product. Change the 2-bars for two 3-bars. Then there is 3 x 3, which is the square -9; we have it there.
Continue changing the beads following the same order as the angular presentation that preceded. After we have reached 4 x 4 (see diagram) the changing pattern of colors begins to appear. The strip of yellow outlines two sides of the square, which as before becomes bigger and bigger. After four we're going to make the right angle blue. Continue with 5 up to 10, where the angle is gold.

Part Two-Building the Tower
The square has changed its colors, but we haven't finished here. We can see that the diagonal is made up of several squares. The diagonal is like the spinal cord of the square, because, like your spinal cord, it supports everything and keeps it straight.

(Note: present one of these methods to the children)

1st Method: 1 x 1 = 1 It remains the same. Look at the green angle. We have 2 x 2 and 1 x 2. When we place these 2-bars together we make a square of 2. Exchange these 2 bars for a square of 2, since there is no room for it in the places vacated by the 2-bars.
Starting at the column of three, slide the bars of 33 x 3 towards the bar of 3+ (3 x 1) to see that when combined they form a square of three. Substitute the bars for a square which is placed in the vacant column. Going to the row of three, do the same3 x 2 + 3 x 1 = a square of three.
Continue with the column of four. Push one bar from 4 x 2 down to meet 4 x 3:
4 x 3 + 4 x 1 forms a square. Replace these bars with a square. On the row repeat the procedure: 4 x 3...+...4 x 1...forms a square. Replace the bars with a square. At the head of the row and the column combine the single bars to form 4 x 2 and 4 x 2. Combine these two groups to make a square which is placed on top of the one on the diagonal.
(cont.)
Continue with the column of five: 5 x 4 + 5 x 1 a square. Repeat on the row. The on the column: 5 x 2 + 5 x 25 x 1 = a square. Repeat on the row.
As a result the square will be transformed into an uneven arrangement of squares on the diagonal, with 2 squares stacked at each even number. Superimpose all of the squares to make an oblique line of stacks. These are then transformed into cubes and stacked to make a tower.
Our spinal column has been transformed into a tower, just like the pink tower. Everything that was spread out on this table has been used to construct the tower. Our Pythagorean table has been transformed into a beautiful multi-colored tower just like the ugly duckling became a swan.

Variation on the 1st Method
Since the newly obtained square of two will not fit in the vacated places, we can move the original square and place one square in the column and one in the row. When the problem arises at each even number, re-arrange the portions of the square slightly so that half are in the column and half are in the row. The result will be a broken diagonal, having a vacant space at each even number.

2nd Method
As before, 1 x 1 is left alone, and 2 x 1 is combined with 2 x 1 to form a square which is stacked on top of the other square. The combination of groups of bead bars will always be with reference to the existing square.
For three, choose the bar on the column which is farthest from the square and combine it with the group of bars on the row which are nearest to the square: 3 x 1 + 3 x 2 = a square. Place the new square on top. Continue combining: 3 x 2+3 x 1
gives a square. The third square completes the stack.
For four, start with the bar on the column furthest from the square, combining it with the group of bars on the row nearest the square4 x 1 + 4 x 3 = a square. Then 4 x 2...+...4 x 2... = a square. Continue until all the squares are stacked forming an oblique line. Proceed as before.

Part Three-Decomposition of the Tower
This collective activity is similar to the bank game for the decimal system. One child acts as the banker, while the others change the quantities.
Begin by dismantling the tower of cubes to make a diagonal of cubes. Starting with two, the child recalls that the cube is made from 2 squares of two. The cube is replaced by a stack of squares. Continue until all of the cubes are transformed into stacks.
Take one square of two and ask, 'What is the square made of?' two 2-bars. Exchange the square. Where will we put them? They correspond to 2x1 and 1x2; therefore place them accordingly. Continue to break down squares in a way that is the opposite of the way you chose to construct them. Notice again the angles of color being formed, with the square always serving as the vertex. The result will be the original Pythagorean table from which we began.
This activity is complicated only in the sense that the tasks must be divided among the numbers of the group.

Note: (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10)2 = 552 = 3025
13 + 23 +33 +43 +53 +63 +73 +83 +93 +103 = 3025

SKIP COUNT CHAINS - FURTHER EXPLORATION

c. <u>Decanomial (a polynomial having ten terms) & The Construction of Chart I</u>

Numeric Decanomial

Materials:
...a control chart
...envelopes numbered 0-9: 0 contains 10 blue squares, 1-9: rectangular pieces on which products are written (i.e. 5 has products of
5 x 6, 5 x 7, 5 x 8, 5 x 9, 5 x 10, 6 x 5, 7 x 5, 8 x 5, 9 x 5, 10 x 5)

Presentation:
Take out the contents of envelope 0 and lay them out in order on the diagonal. If the child hasn't had powers, simply explain that 102 is one way we can write 10 x 10.
Examine the contents of envelope 1, forming pairs of like rectangles. Mix them up and fish for one. The child reads the product, thinks of the combination and places the rectangle in the formation, just as in the Bingo Game. Allow the child to continue choosing envelopes in any order. At a later stage, eliminate the envelopes 1-9 and mix the pieces in a basket from which the child fishes.
Later the child will realize that this puzzle like the table of Pythagorean is symmetrical, having equal products on opposite sides of the diagonal.

Age: between 6 1/2 and 81/2 years

Aim:
...to incarnate the geometric figures formed by multiplication
...an indirect preparation for square roots and the square of polynomials

SPECIAL CASES

Materials:
...multiplication combination booklet
...chart
...black and red pens
...special combination cards

Presentation:
The procedure is as for special cases in addition and subtraction, resulting in a chart as below: (The words in parentheses do not appear on the chart, but are given orally)

0- Calculate the product
2 x 3 = ? (two taken three times; what does it give me?)

1- Calculate the Multiplier
2 x ? = 6 (two taken how many times will give me six?)

2- Calculate the Multiplicand
? x 3 = 6 (what number taken three times will give me six?)

3- Inverse of Case Zero---Calculate the Product
? = 2 x 3 (what number do I obtain when I take two, three times?)

4- Inverse of The First---Case, Calculate the Multiplier
6 = 2 x ? (six is equal to two taken how many times?)

5- Inverse of the Second Case ---Calculate the Multiplicand
6 = ? x 3 (six is equal to what number taken three times?)

6- Calculate the Multiplier and the Multiplicand
6 = ? x ? (six is the number I obtain when I take a certain number, a certain number of times; what are these numbers?)

Note: In cases 1, 4 and 7, the child performs multiplication, but in all others division is indirectly involved

Activity:
The seven special combination cards are left at the disposition of the children, combining these with the cards given previously for special cases of addition and subtraction.

Aim: further understanding of the concept of multiplication

THE BANK GAME

This activity is the culmination of the many skills the child has developed: addition, multiplication at the level of memorization, multiplication and division by powers of 10 and changing from one hierarchy to another. This work is parallel to the large bead frame.

Materials:
...box containing 9 series of white cards - product
...4 series of colored cards - multiplication
...3 series of gray cards - multiplier
...box of signs for the operations

Presentation:
Invite the children to lay the cards out in columns, naming them as they go. The child should recognize the numerals up to 9 million. If necessary identify for him 10 million and 100 million. One child must go to the bank. (For the first demonstration use a one-digit multiplier. In all other work a two digit multiplier will be used) Write the problem on a piece of paper. The child takes this to the bank and sets up the multiplication, using colored cards for the multiplicand, signs for the operation and a gray card for the multiplier.

4876 x 6 =

We then decompose the multiplicand, just as was done on the bead frame for multiplication, and begins multiplying:

			6	x	6	=
		7	0			
	8	0	0			
4	0	0	0			

6 x 6 = 36. He asks the banker for 36. The two cards are placed separately at the near edge of the table. Move the cards for the operation and the multiplier to align them with the next digit - 70. 7 x 6 = 42, 42 tens. He asks the banker for 420. There are placed with the previous product cards so that columns are being formed.

	20	
400	30	6

The operation continues in this way. When all of the digits have been multiplied the children-assembles the multiplicand. To find the product, he begins with the lowest hierarchy, combining and making changes. The product cards are assembled and placed by the equal sign. The children record the equation.

Let's try a different one. Write the problem on a piece of paper:
6835 x 48 =. The child sets up the problem with the colored cards and the gray cards. We don't have a gray card for forty, so we place a 4 next to a 0 to make 40, then place the 8 on top of the 0 to make 48.
As before, the child decomposes the multiplicand. We only want to multiply by one digit of the multiplier at a time. Remove the 40 and set it aside, yet together for later retrieval
Begin multiplying as before: 5 x 8 = 40. The banker gives 40, etc. After the multiplicand has been multiplied by the units of the multiplier, we can begin multiplying by the tens. (8 and 40 switch places, 8 is turned over) However, just as with the bead frame we have the rule: the multiplier must be units. Transfer the zero from the multiplier to the multiplicand.
Continue multiplying. The child will realize that 8 x 4 = 32 and 3 zeros after it- makes 32,000. In the end make changes in the product, carrying mentally, The product is assembled, the equation is recorded.

Notes: Since there is only one set of cards for the product much changing will be involved, calling for quick mental addition and subtraction at the level of memorization, on the part of the banker.
In multiplication with a two-digit multiplier, the child never stops in this game to record partial products, because the aim here is to develop agility in changing from one hierarchy to another. With only one set of cards it would be difficult to set aside partial products.
It is interesting to perform the same problem using the bank game, the checkerboard and the large bead frame to recognize the similarities and differences in the work. Be aware of the limits of the bead frame.

Age: 7 years

Aim: development of mental flexibility to prepare for mental calculating

Word Problems

As for addition and subtraction, word problems are prepared on cards which deal with each of the seven special cases. These are mixed in with those given for addition and subtraction.

Example:
Steve has 48 stamps in his collection at school. Each day he brought a certain number of stamps, for

a certain number of days. How many stamps did he bring each day? and how many days did he bring stamps?

The child writes his answer:

48 = 6 x 8
48 = 8 x 6

Subtraction

Stamp Game Subtraction

Dynamic Subtraction

Materials:
...wooden stamps of four types:
...green unit stamps printed with the numeral 1,
...blue tens stamps printed with the numeral 10,
...red hundred stamps printed with the numeral 100, and
...green thousand stamps printed with the numeral 1000
...box with three compartments each containing 9 skittles and one counter in each of the hierarchic colors;
...four small plates

Presentation:
Again using the work cards, the child does a subtraction. The first number is formed, and from this number a second quantity is taken away and assembled in columns well below the first group. When the child sees that a change is necessary, the stamp of the higher order is placed between the rows as ten of the lower order are counted out. The ten are then placed in the row, the other stamp is put back in the box. When the subtraction is complete the child reads the result-what was left behind. The problem is recorded in his notebook
Would you like to see how we can check to see if you did the subtraction correctly? We can put these two quantities back together. The stamps are counted and changed. 'Did you get your original number?' That means that the subtraction was done correctly.
The child should also have practice subtracting with zeros in the subtrahend.

ex. 4000 - 785 =

Memorization Exercises

STRIP BOARD

a. Introduction and List of Materials

Introduction:
The child first dealt with the concept of subtraction with number rods. Later he learned the concept with the decimal system material, and the stamp game. Through memorization the child will master all of the combinations necessary for his work.

Materials:
...Subtraction Strip Board, which differs from the addition strip board in that the numerals 1-9 are in

blue,
followed by a blue line, and 10-18 in red.
...Box of 17 neutral strips (to limit the minuend) 9 blue strips (to function as the subtrahend) and 9 sectional pink strips (to serve as the difference)
...Booklet of Combinations (page one deals with 18)
...Box of Subtraction Combinations (same combinations as are found in the booklet)
...Box of blue tiles for bingo game
...Subtraction Charts I, II, III (for control)

STRIP BOARD

b. Initial Presentation

To familiarize the child with the subtraction strip board, the teacher demonstrates. The neutral strips and the blue strips are lain out in the pipe organ arrangement. The teacher chooses a neutral strip. This is used to cover the numerals we don't need
The child then chooses a number to subtract, i.e. 5 The blue 5 strip is placed end to end with the neutral strip. The answer for 13-5 is the first number that shows....8. If by chance the child chooses a subtrahend that would give a difference greater than nine, the teacher explains that the maximum difference that can exist is nine, 13 - 2 for example is not necessary.

STRIP BOARD

c. Subtraction Booklets

Materials:
...booklets of 18 pages; each page has combinations with a common minuend- the first page deals with 18
...Subtraction Strip Board, strips
...Chart I

Exercise:
The child begins with the first page in his booklet. With 18, a neutral strip is not needed, it is already the last number in the row. The combination is 18 - 9; therefore, the blue strip for nine is placed over the numbers. The first number to show is 9. That's the difference, and it is written in the booklet. 18 - 9 is the only combination possible. The child may try others to prove this.
Going on to the second page, the minuend is now 17, therefore the smallest neutral strip is used to cover 18 (the number greater than 17) The child reads the first combination, takes the blue strip corresponding to the subtrahend and places it over the numbers. The answer is read and is written on the form. For the second combination, we know that the minuend will be the same, therefore the neutral strip doesn't need to be changed.
After completing the page, the child should notice 17 - 9 = 8 that while the minuend is fixed, there is 17 - 8 = 9 a decrease of one unit in the subtrahend, and, therefore, an increase of one unit in the difference.

Notes: The blue strip is used as the subtrahend because the child must realize that he is subtracting a group. In subtraction, the aim is always to break down the ten. Since the neutral strips occupy much space, after this exercise the child may use only the longest, sliding it off the edge to its correct position.

Observations on the Subtraction Chart I:
This chart reproduces all of the combinations in the subtraction booklet. In the first 9 columns the differences are common in horizontal rows. This indirectly shows the invariable property of

subtraction; if one adds a number to both the subtrahend and the minuend, the difference is the same, i.e. 1 1 = 0, 2 2 = 0, 3 3 = 09 9 = 0
In the last nine columns, the subtrahend is consistent in each horizontal row; thus the minuend and the difference increased by one, i.e. 10 9 = 1, 11 9 = 218 9 = 9

STRIP BOARD

d. Combination Cards

Materials:
...subtraction strip board, strips (neutral and blue)
...combination cards
...Chart I (for control)

Presentation:
The child fishes for a combination, reads it and writes it on his paper, 15 7 =. The neutral strip is used to cover all of the numbers greater than 15 which are not needed. Next to the neutral strip is placed the blue 7 strip. The difference is the last number showing; this is recorded.
The exercise continues as long as the child would like, and then he controls his work with Chart I.

STRIP BOARD

e. Decomposition of a Number

Materials:
...subtraction strip board, all of the strips

Presentation:
(In this exercise the pink strips are used for the first time to function as the difference)
Let's see how many ways we can decompose (break down) 9? Nine will be the minuend, therefore a neutral strip is placed over the number greater than 9. The teacher writes down the combination 9 1 =___. (Note: decomposition always begins at one, removing one unit at a time) The subtrahend is one, so a blue 1 strip is needed. This time it is placed on the first row, under 9. The child guesses the answer and tries to place the pink strip for his answer on the row. If it fits he knows that he is correct. The answer is recorded. The work continues in order until all of the blue strips are used, and a column of combinations has been completed9 9 = 0
In this exercise the child may recall his work of this fashion in addition, which resulted in elimination of some combinations. In subtraction, all of the combinations are needed and must be learned.
The child tries to decompose other numbers in the same way: i.e. 14. How many ways can 14 be decomposed? The neutral strip identifies 14 as the minuend. Can I do 14 1? No (the one strip may be tried, but it will not work because 9 is the maximum difference we can have) 14 2? 14 4? 14 4? 14 5? Yes. The decomposition begins here laying out the blue 5 strip and the pink 9, recording 14 5 = 9, and so on to 14 9 = 5.

Control of Error: Chart I

STRIP BOARD

f. Decomposition of a Number with Zero as the Subtrahend

Materials:
...subtraction strip board, all of the strips

Presentation:
As before, the teacher presents a number to decompose. The neutral strip is lain over the number to limit the minuend. On a piece of paper, the teacher writes, i.e.
7 0 =___. What must be taken away? nothing. In subtraction also, we see that zero doesn't change anything. On the first row, then, the pink strip for seven is placed and this difference is recorded. 7 0 = 7 The child continues7 7 = 0

Control of Error: Chart I

THE SNAKE GAME

Materials: same materials as for previous snake games:
...box of colored bead bars 1-9
...box of ten bars
...box of black and white reminder bars (place holders)
...also box, with 9 compartments, for gray bead bars 1-9
(for bars of 6-9, there is a small space or color change after the fifth bead to facilitate counting)

Presentation:
As before, a snake is made, though this time we add to these colored bead bars, some gray bead bars. (Note: before the first gray bar appears, several colored bead bars should appear to create a large minuend) As before we begin counting, using the black and white reminder bead bars. When we come to a gray bar, we must subtract. The preceding black and white bead bar and the gray bar are isolated. 8 4 = 4 The 4 black bar is placed in the box cover with the other original colored beads.
Counting continues as usual. The next two bead bars are isolated: black 3 and gray 7. This gray bar means I must subtract. 3 7 is impossible; therefore I take one ten bar (from the snake's new skin) and place it beside the 3 to make 13. 13 7 = 6. The black and white 6 is placed in the snake; the black 3 is placed with the other reminder bars; the ten bar is placed back in the box with the other ten bars; the gray 7 is placed in the box cover with the other original colored bead bars.
When the counting is finished, the gold bead bars (and reminder bar) are counted to find the result.

Control of Error: In the box cover are colored bead bars and gray bead bars mixed. First, these are separated into two groups, lain in chronological order. A gray bar is placed with its equivalent colored bar. Two colored bars may be combined to match a gray bar, or vice versa. When all of the gray bars have been matched; the colored bars are paired to be matched with ten bars as usual . When all of the bars have been matched, we know the counting was done correctly.

Direct Aim: to memorize subtraction

Indirect Aim: to prepare for algebra: positive and negative numbers

SUBTRACTION CHARTS AND COMBINATION CARD EXERCISES

a. <u>Passage From Chart I to Chart II</u>

Materials:
...Chart II (the numbers in pink function as the minuend; the blue as the subtrahend)
...combination cards
...Chart I

Exercise:
The child fishes for a combination, i.e. 9 2 =___, reads it and writes it down on his paper. A finger

is placed on 9 on the pink strip on the chart.; another finger is placed on 2 on the blue strip. Where the two fingers meet, we find the difference. This is recorded on the paper. The child continues his work in this way, and when finished, he controls with Chart I.

SUBTRACTION CHARTS AND COMBINATION CARD EXERCISES

b. The Bingo Game of Subtraction (using Chart III)

Materials:
...Chart III and box of corresponding tiles
...combination cards
...Chart I (for control of combinations)
...Chart II (for control of placement of the tiles)

A. Exercise:
The tiles are randomly arranged face up on the table. The child fishes for a combination, thinks of the answer, and finds the corresponding tile. After the minuend (pink) and subtrahend (blue) have been established on the chart, the child is able to find the place for the tile. He writes the equation on his paper and continues.

Control of Error: Charts I and II

B. Exercise:
All of the tiles are in the box (or in a sack). The child fishes for a tile, and thinks, what could this be the remainder of? He thinks of a combination and writes it down, i.e.
7 = 14 7. He puts the tile in its place. He continues in this way, then controls his work.

C. Exercise:
The tiles are arranged on the table in common stacks. The child chooses one stack and thinks of combinations which will yield this difference. He writes down the combination, finds the place on the chart, and so on, continuing until he has finished the stack.
When all of the stacks are arranged in order in a row, what form do they make? a rectangle or parallelopiped.

Group Games
1. The teacher, or a child functioning as the teacher, fishes for a combination and reads it. One of the children guess the difference.

2. The teacher fishes for a tile and the children offer combinations which give that result, until all are given.

Aim: (of all of exercises,) to memorize subtraction combinations

Bead Frame Subtraction

STATIC SUBTRACTION

Presentation:
The teacher initially presents a static subtraction. The problem is written on the form and the

minuend is formed on the frame. Now we must take away this quantity. Beginning with the units move the beads back to the left. Continue with tens, hundreds and thousands. Read the result and record it.

DYNAMIC SUBTRACTION

A second (and all the others following) example is dynamic:
Again the minuend is formed. Beginning with the units we must take away this quantity. Begin counting the units as they are moved back...1,2,3,4,5 we must change one ten to ten units (move one ten back, 10 units forward) and continue...6,7 moving the unit beads back. Subtract tens, hundreds and thousands in the same way. Read and record the result.

CALCULATING THE DIFFERENCE A LITTLE MORE ABSTRACTLY

Presentation:
With a dynamic subtraction again, the child forms the minuend. As before begin by taking away the units. Record the number of units remaining, in the units column. Subtract the tens and so on. The result on the form will correspond to the difference formed on the frame.

Games:
Form 1000 on the frame. Subtract one unit. Form 1000 on the frame and subtract 999. Each time three changes are necessary.

More Memorization Exercises (Special Cases)

Note: As in the special cases for addition, these exercises are done after the child has already done much work with the previous exercises.

Materials:
...subtraction combination booklet
...large sheet of paper or chart,
...red and black pens
...special combination cards

Presentation:
The teacher opens to a page, in the booklet. When you do these combinations, what do you do? calculate the difference.
As before the teacher writes the tile on the chart, gives an example and reads. In all of the other cases, the special case is set up first in the combination booklet or on a sheet of paper, and the child identifies the missing part. Then the special case is added to the chart.

0- Calculating the Difference
16 9 = ? (If from 16, I take away 9, what will be left?)

1- Calculating the Subtrahend
16 ? = 7 (From 16 I have taken away a certain number, and 7 is left. How much did I take away?

2- Calculating the Minuend
? 9 = 7 (From a certain number I have taken away 9, and 7 is left. What was that number?

3- Inverse of Case Zero - Calculating the Difference
? = 16 9 (What will be left if from 16 I take 9?)

4- Inverse of the 1st Case- Calculating the Subtrahend
7 = 16 ? (7 is the remainder when from 16 I take away what number?)

5- Inverse of the 2nd Case- Calculating the Minuend
7 = ? 9 (7 is the remainder when from a certain number I take away 9. What is that number?)

6- Calculating the Minuend and the Subtrahend
7 = ? ? (7 is the remainder when from a certain number I take away another number. What is the first number, and what is the second number?)

Collective Activity:
The teacher passes out the special combination cards. Each child reads the combination, states what part is missing and which case it is. (referring to the chart)

Individual Activity:
The child works with the special combination cards as before: drawing one and computing it, then writing the equation in his notebook with the answer in red.

Aims: further memorization of subtraction combinations
reinforcement and understanding of the concept

Notes: This activity develops the ability to work with the same combination under different conditions and from various points of view, thus creating flexibility in the child's mind. Here, also, we see the close relationship between addition and subtraction.

Word Problems

E. Word Problems

As before word problems are prepared dealing with the seven different cases. These 7 cards are mixed with the 7 addition cards. When the child has done all of these, he may invent his own which will indicate his understanding of these special cases.

Example:
1. Rebecca has 6 roasted chestnuts left. Before she had 11. How many did she eat? 6 = 11 5

Age: between 6 and 7 years

Division

Stamp Game Division - Single Digit

STAMP GAME DIVISION - SINGLE DIGIT

Materials:
...wooden stamps of four types:
   ...green unit stamps printed with the numeral 1,
   ...blue tens stamps printed with the numeral 10,
   ...red hundred stamps printed with the numeral 100, and
   ...green thousand stamps printed with the numeral 1000
...box with three compartments each containing 9 skittles and one counter in each of the hierarchic colors;
...four small plates

Presentation:
The child is given a division problem on a work card. Since this is division, we put the stamps in these little dishes. 'How many must this quantity be distributed to?' Instead of calling children we can use these green skittles, one skittle for each child . Now you begin the division.
The skittles are set out in a row. Beginning with thousands, the child distributes equally to the skittle. When the child runs out of i.e. hundreds before each skittle has received one, the hundreds are returned to the dish, and the child goes on to tens. If he doesn't have enough tens (or if he runs out) one of the remaining hundreds is changed to ten tens. This continues until all of the hundreds have been changed.
When the child has finished distributing, he reads the result-what one skittle received. The problem is recorded in his notebook. Note: If necessary to simplify counting, the stamps of one skittle may be arranged in hierarchic order after the distribution is finished.

Age: 6-7

Aims: to perform the operations at a more abstract level (here there is no visual difference in value according to size)

Note: These materials are used parallel to operations in the decimal system

Memorization Exercises

DIVISION BOARDS - BEAD BOARDS

a. Introduction and List of Materials

Introduction:
The memorization of division is the synthesis of the four operations. For this reason the child must precede this work with a great deal of work with the other operations, especially multiplication. It is very important that the child know multiplication really well before going on to division.
The child has encountered division before via many materials and regarding many cases: distributive and group division, division with a 1 or 2 digit divisor. In order to go on, the child must memorize certain combinations in division.

Materials:
...Division Bead Board: the numerals 1-9 across the top on a green background represent the divisors; the numerals 1-9 down
   the left side represent the quotients; the 1 in the green circle in the upper left indicates that the numbers 1-9 below it represent
   units; 81 holes.
...Orange box containing 9 green skittles
...Orange box containing 81 green beads
...forms

...Booklet of Combinations (36 pages)
...Box of Multiplication Combinations (same combinations as are found in the booklet)
...Box of orange tiles for bingo game
...Charts I, II (for control)

DIVISION BOARDS - BEAD BOARDS

b. Initial Presentation

Preparation for Presentation
Before-hand, the teacher prepares pads of problems which consist of 81 forms. On each form, in the spaces at the top, is written a number, which is the constant dividend for that form. The forms are arranged on the strip in order according to the dividends, starting with 81, ending with 1. Under the forms on which a combination with remainders appears, the ordinal number of the form is written in red. On each form are written all of the combinations possible to perform on the bead board, beginning always with 9 (except, of course, when the dividend is less than 9) All of the combinations which have a remainder of zero are underlined in red.

Presentation
In this box are 81 beads which we will distribute to these 9 skittles. The nine skittles are placed along the top green strip of numbers 1-9. These 81 beads are the dividend, so we write 81 under the column 'dividend.' Then write the sign for division. The beads must be divided among 9 skittles; nine is our divisor, so we write 9 under 'divisor.' Now give out the beads in rows until all the beads are given out equally.
Each skittle received nine beads, (note the 9 at the end of the row), so we write 9 under 'quotient.' The last column is for the remainder, but here we have no remainder, so we write zero in that column. Whenever we have a combination that has no remainder, it is very important for our work, so we underline 'it' in red.
Let's try 81 ÷ 8. Remove one skittle and the extra beads to their respective boxes. What is the quotient and the remainder? 81 ÷ 8=9 r 9 In this game there are two rules to be followed:
1) The quotient should never be greater than 9.
2) The remainder may never be equal or greater than the divisor.
Therefore, we cannot have this combination because the remainder is too big.
Go on to 80. Change to a new form. Remove one bead and place it back in a box. Start with 80 ÷ 9=. Write the combination on the form, distribute the beads and count the remainder. 80 ÷ 9 = 8 r 8
Try 80 ÷ 8. Remove one skittle, redistribute the beads, and count the remainder.
80 ÷ 8 = 9 r 8. This cannot be used because the remainder is too big, being equal to the divisor. Erase or cross out this combination.
Go on this way until 72 so that the children see another page on which combinations can be underlined. At this point bring out the prepared roll of forms. On this strip we can see all the combinations that have zero remainder. All of the forms that have at least one combination with remainder zero, have been reproduced into booklet form for your work.

Aim: to understand how the combination booklet was formed

DIVISION BOARDS - BEAD BOARDS

c. Division Booklets

Materials:
...Division Bead Board
...box with beads
...box of skittles
...Combination booklet
...Chart I

Exercise:
The child chooses a form in the booklet, i.e. 7 Since 7 is the dividend, count out 7 beads into the box cover. The first combination is 7 ÷ 7 =, so 7 skittles are put out; the 7 beads are divided among them. Each skittle receives one; 1 is the quotient. There is no remainder. 7 ÷ 7 = 1 r 0
The next combination is 7 ÷ 6 =. Only six skittles are needed. Each skittle receives one and the remaining bead is placed in the bottom row, or in a box cover. 7 ÷ 6 = 1 r 1. Continue until the form is completed.

Control of error: Chart I. Find the dividend along the top and the divisor in red at the left. Go down and across to find the quotient where the fingers met. If there is a remainder, the box will be empty, thus move along the row to the right until a box is full. There you will find the quotient. To check the remainder subtract the dividend at the top of that column, from your original dividend. The difference is your remainder.

Note: When presenting the chart to the children, we identify the prime numbers as well, since they are shaded in red. 7,5,3,2 and 1 are special numbers because they only have as divisors, themselves and one. 7 can only be divided by 7 and by 1, and so on. These special numbers are called prime numbers.

DIVISION CHARTS AND COMBINATION CARDS

a. Chart I

Materials:
...box of loose combination cards, only those that have a remainder of zero
...Chart I (as a control)

Exercise:
In this box are only the even division combinations: those having a remainder of zero. The child fishes for a combination, reads it and copies it into his notebook. On the chart he finds the dividend at the top and the divisor on the left side. The place where the two fingers meet is where the answer is found. He writes the quotient to complete the equation.
Later the child can do the combinations in his head, write down the quotient and use Chart 1 only as a control.

DIVISION CHARTS AND COMBINATION CARDS

b. Bingo Game (Chart II)

Materials:
...box of tiles
...box of combination cards
...Chart II and Chart I (for control)

Note: Much practice should have preceded these exercises.

Exercise A:
Spread out the tiles face up. The child fishes for a combination, writes it down including the quotient, and finds the corresponding tile. The tile is placed on Chart II appropriately.

Exercise B.
With all of the tiles in the box, the child fishes for a tile. He thinks of a combination that would yield that quotient and writes the equation in his book, i.e. 8 = 56 ÷ 7. The tile is placed on Chart II appropriately.

Exercise C.
All the tiles are stacked as usual (this time forming a parallelopiped) The child chooses a stack, and one at a time he thinks of all the possible combinations that will yield that quotient, writes them down and places the tiles on Chart II appropriately. This continues until all the tiles of the stack which was chosen, are used. The child uses Chart I to check if he found all of the possible combinations and if they were placed correctly.

Note: For many children the aim of this work can be to fill up the entire board.

Group Game 1.
The teacher or a child leading a group of children draws a combination and reads it. One child responds.

Group Game 2.
The teacher draws a tile and reads the quotient. One child may try to give all of the possible combinations, or each child in turn may give one until all of the possibilities are exhausted.

Division By More Than One Digit

DOUBLE DIGIT DIVISION

a. Intro. to Double Digit Division (on The Change Game)

Materials:
...golden bead materials and numeral cards
...ribbons: green, blue, (also red for later)

Presentation:
The teacher prepares the numeral cards and asks a child to bring the corresponding quantity. Now I would like to distribute this quantity among twelve children. Twelve children are called, but this creates such confusion. How can we solve this problem? Twelve is made up of ten and two units. Two children can represent the two units and can be given green ribbons. These ten children must choose one who will represent ten. What color ribbon do we give the representative?
Now we are ready to distribute this quantity. If we give one thousand to the child who represents ten children, what will each of the other children get? 100 (because 1000 is 10 hundreds) also each of these children receives 100another thousand for the group of ten, another hundred for this one, and another hundred for this one. This continues, making all of the necessary changes, until all has been distributed (perhaps leaving a remainder)
What is our result? The result is what one unit receives. This child who represents ten children has enough on his tray so that each of these ten children will receive what one child received. (This quantity may be distributed if necessary.)

Control of error: The quantity may be added together again, making the necessary changes to form the original number.

Age: 6-7

Aim: to learn the concept of two-digit division: that if the ten receives a certain quantity, the unit receives 1/10 of that quantity.

DOUBLE DIGIT DIVISION

b. Double Digit Division on The Stamp Game

Materials: stamp game work cards

Presentation:
The first quantity for this problem is formed with the stamps and placed in little dishes. We need to divide this among twelve. This blue skittle can be used to represent 10, and these green skittles will represent the two units. Now we must give out this quantity. One thousand is given to the ten, so how much does each unit receive? 100. The child distributes and changes as necessary, 'one hundred to the ten, ten to the units, another hundred to the tenand so on.' What did one receive? The child records the problem in his notebook.

Control of error: The quantity may combine the quantities distributed, count them, change them, to obtain the original number.

Age: 6-7

Aim: to practice two-digit division

TRIPLE DIGIT DIVISION

a. Intro. to Triple Digit Division (on The Change Game)

Materials:
...golden bead materials and numeral cards
...ribbons: green, blue, (also red for later)

Presentation:
Given a division problem, the child can see that it would be impractical to distribute one by one to over a hundred people. The child representing 100 wears a red ribbon. When red (100) receives 1000, blue (ten) will receive 100, and units (green) will receive 10.

Control of error: The quantity may be added together again, making the necessary changes to form the original number.

Age: 6-7

Aim: to learn the concept of three-digit division.

TRIPLE DIGIT DIVISION

b. Triple Digit Division on The Stamp Game

As in the previous presentation, skittles are used. Here one red skittle represents 100.

Division Involving Zeroes

DIVISION INVOLVING ZEROS

Materials: stamp game work cards

Presentation:
When the child places the stamps in the dishes to show the dividend, the hierarchy that has zero is indicated by an empty dish. When the child needs to change to that hierarchy, the procedure is the same.

DIVISION INVOLVING ZEROS

Materials:
...stamp game
...work cards

Presentation:
The child lays out the first quantity in the dishes. This quantity must be divided among 104. I don't need any tens skittles, but so as not to forget the tens, we place a blue counter in its place. The child begins distributing: 1000 to the hundred, the ten would get 100, but I give it nothing, and the unit received 10and so on.
By this time the child should realize that any remainder must be less that the divisor.

Group Division

SINGLE DIGIT GROUP DIVISION

Materials:
...stamp game
...work cards (divisor up to 9)

Presentation:
The division problem says 24 divided by 4 =____. How many groups of four are we able to make with this number. The four skittles are placed in a bunch and groups of four units are in rows in front of this group. Changes are made as necessary. When the distribution is complete, how many groups of four were we able to make from 24? The rows are counted. Six is the result of this group division. The child then does the same problem using distributive division to see that the result is the same.

Age: 6-7

Aim: to understand a different aspect of division

Note: This is presented parallel to abstract division. Memorization has begun.

DOUBLE DIGIT GROUP DIVISION

Materials:
...stamp game
...work cards

Presentation:

1. Given a division problem, the child prepares the stamps and the skittles. Since we want to do a group division, we put the skittles together in a group. How many groups of 26 can be made from this number? The child places two tens and six units in a row, continuing his distribution by making all horizontal rows of 26 in a column. (The stamps are always placed in hierarchic order) Here, the skittles serve only as a reminder of the number in the group.

2. This time we will first make groups only of tens. Groups are made of two tens and lain out in rows. How many groups of ten did I make? So that each group has 26, I must make the same number of groups of units. Groups of six units are made in rows that line up with the groups of ten, yet in a separate column. When the child finds that more units are needed, one group of tens is returned to the dish so that they may be changed. How many groups of tens do I have ? 4 How many groups of units? 4 is our answer.

Age: 7

Aims:
...to learn the concept of group division
...to continue towards further abstraction in division

More Memorization Exercises

SPECIAL CASES

As for the other operations, we examine the special cases using as a starting point the combination that is most familiar. A chart will be constructed as follows:

0- Calculate the quotient
72 ÷ 9 = ? (72 divided by 9 gives me what number?)

1- Calculate the Divisor
72 ÷ ? = 8 (72 divided by what number gives me 8?)

2- Calculate the Dividend
? ÷ 9 = 8 (What number when divided by 9 gives me 8?)

3- Inverse of Case Zero---Calculate the Quotient
? = 72 ÷ 9 (what number will we obtain by dividing 72 by 9)?

4- Inverse of The First---Case, Calculate the Divisor
8 = 72 ÷ ? (We obtain 8 as a quotient when dividing 72 by what number?)

5- Inverse of the Second Case ---Calculate the Dividend
8 = ? ÷ 9 (We obtain 8 as a quotient when dividing what number by 9?)

6- Calculate the Divisor and the Dividend
8 = ? ÷ ? (We obtain 8 by dividing a certain number by another number. What is the first number and the second number?)

Note: Here we also see the relationship between multiplication and division. In cases 2 and 5 the child must multiply to find the dividend.

SEARCH FOR QUOTIENTS

Materials:
...Chart II
...bingo tiles for multiplication

Presentation:
Have the child find one bingo tile to match all the dividends along the top of Chart II. These are placed in a box cover or something.
The child fishes for a tile, i.e., 24. Let's try to find all the quotients with zero remainders that can be made with this dividend. Start with 24 ÷ 9 =. It won't work so leave it blank, and go on24 ÷ 8 = 3 and so on with the child giving the correct quotients. At the end erase those that would not yield zero remainders, thus leaving space to correspond with Chart I. Notice how the column of quotients matches the column under 24 on the chart.
These are the combinations I wanted, because now we can do 3 x 8 = 24. Write this to the right of 24 ÷ 8 = 3. 24 was my dividend: now it is my product. Go on in the same way for the other combinations making a second column.
The child will realize that if 24 ÷ 8 = 3, then 24 ÷ 3 = 8. It is a sort of game where the numbers change positions.

Aims:
...to find quotients with zero remainders
...to realize the relationship between multiplication and division memorization of division

Indirect Aim: indirect preparation for the divisibility of numbers

PRIME NUMBERS

Materials: Chart I

Presentation:
Recall the child's attention to the numbers in pink on the chart, which were called prime numbers, and which can be divided only by themselves and 1. These are very important numbers because they form all of the other numbers. We can see that this is true. (refer to the chart) 1, 2, and 3 are prime numbers. 4 is not a prime number, but is made of 2 x 2, and 2 is a prime number. Go on to 6 which is not prime. It is made up of 3 x 2; 3 and 2 are prime numbers.
If you try to decompose any number, you will find that it is made up of prime numbers. Invite the child to choose one of the dividends, and think of one combination: 24 = 3 x 8. 3 is a prime number,

but not 8, 8 is made up of 2 x 4. 2 is a prime number, but not 4, 4 is made up of 2 x 2. 2 is a prime number.

Try another combination of 24 to check.
24 = 6 x 4 neither 6 nor four is prime
6 = 2 x 3 both are prime
4 = 2 x 2 both are prime

Direct Aim: to realize the importance of prime numbers

Indirect Aim:
...to prepare for divisibility of numbers, LCM-least common multiple GCD-greatest common divisor and
reduction of fractions to lowest terms

Aims:
...(for all of division) to memorize the combinations necessary for division.
...to stimulate an interest that will help him to use the experiences acquired previously.

Division With Racks & Tubes

INTRODUCTION

The operations of addition, subtraction and multiplication can be performed with the large bead frame. But since the beads are connected to the wires, the beads frame cannot be used for division. The hierarchic material for division consists of loose beads.
Up to this point division has been done with the decimal system material to give the concept, including division with a 2-or 3-digit divisor, and group division. These concepts were reinforced with the stamp game. Following a research of the combinations necessary for memorization, where the quotient was limited to a maximum of 9. Division was dealt with indirectly in many of the multiplication activities. Using this material the dividend may have up to 7 digits and a divisor of 1-, 2- or 3-digits may be used.
Maria Montessori referred to this material "as an arithmetical pastime for the child." This work clarifies the analytic procedure for the development of the operation. The fundamental difficulty of division is obtaining the digits of the quotient, recognizing their values and placing them in their proper hierarchical position.
At this level more importance is given to the quotient, that is, what each unit receives, and not so much to the quantity to be divided.

MATERIALS: RACKS & TUBES

Materials:
...7 test tube racks: 3 white, 3 gray and 1 black
...each rack contains 10 test tubes

deposit each tube contains 10 loose beads
[These are the "deposit" from which quantities are drawn. The racks are white for the simple class, gray racks for thousands and 1 black rack of green beads for units of millions.]

7 bowls - 1 to correspond to each rack:
dividend the exterior of the bowl corresponds to the color of the rack the interior of the bowl corresponds to the color of the beads The dividend is formed in these bowls, just as was done with the stamp game.

3 bead boards: in hierarchical colors for units, tens, hundreds
For a 1-digit divisor the green board is used
divisor For a 2-digit divisor the green and blue boards are used
For a 3-digit divisor the green, blue and red boards are used.
[As in memorization, the distribution is done on the boards.]
Box with three compartments containing nine skittles of each of the three hierarchic colors that represent the divisor.

Also: A large tray to hold the racks while they are not in use.

SMALL DIVISION (1-DIGIT DIVISOR, 4-DIGIT DIVIDEND)

1st level

9764 ÷ 4 **=** $4\overline{)9764}$

Isolate the racks that are needed to form the dividend. Place the other racks on the tray. Pour the quantities into the respective bowls. Place the green skittles on the green board for the divisor. Begin distributing "bringing down" the units of thousands, that is moving the rack and bowl closer to the board. After distributing the units of thousands, record the first digit of the quotient in its hierarchical color, reading the number at the left side of the board.
Remove the beads from the board and place them back into the tubes. There is one units of thousand bead remaining in the bowl, which can't be distributed as is. Change it for 10 hundreds (pour the hundreds into the hundreds bowl). Having finished with the units of thousands, place the rack and the bowl out of the way on the tray.
Continue in the same way for hundreds, tens and units.

9764 ÷ 4 = 2441 $\begin{array}{r} 2441 \\ 4\overline{)9764} \end{array}$

Note: Here also the operation is reduced to the level of memorization.

Recall the problem 81 ÷ 8 = which couldn't be done before. This does not mean that it couldn't be done, just that it couldn't be done with that material. Try it using this new material.
It is important to emphasize that every time a hierarchy is considered, a digit must be placed in the quotient. If there are not enough beads to distribute, we must still record a zero. This is where the child could easily make a mistake.

2nd level

9216 ÷ 3 **=** $3\overline{)9216}$

Set up the problem as before and begin distributing the units of thousands. Record the first digit of the quotient using the hierarchical color green. There are no beads in the bowl, so the remainder is zero. Write the remainder under the 9. Put the thousands away.

9216 ÷ 3 **=** $\begin{array}{l} \;\;3 \\ 3\overline{)9216} \end{array}$

0

Bring down the hundreds, that is, move the bowl closer to the board and write the 2 next to the 0. Since we can't distribute these two beads, write the next digit in the quotient and write the remainder. Remove the beads.

```
9216 ÷ 3 =    30
             3|9216
02             02
 2              2
```

The two hundreds must be changed for tens. Put away the hundreds. Bring down the tens. Now we must distribute 21 tens. Continue in this way.

```
9216 ÷ 3 =   3072
             3|9216
02             02
 21             21
  06             06
   0              0
```

Note: It is important to work through a problem such as 1275 ÷ 3 = to demonstrate the grouping of the first two digits. In a case such as this we do not record a zero in the quotient for the first hierarchy

SMALL DIVISION

3rd Level-Group Division

Note: The child will never reach abstraction using the distributive division technique.

Introduction:
Recall the concept of group division. Take out 15 loose golden beads. Invite the child to find out how many groups of three can be formed.
Relate word problems to demonstrate the difference between distributive division and group division:

1. I have 12 pencils which I must give to 6 children. How many pencils will each child get? 2. What kind of division did we do? distributive division.

2. I have 25¢. I want to buy pencils costing 5¢ each. How many pencils may I buy? 5. This time I had to think of how many groups of 5 are in 25. This is *group division*.

Note: The Difference here is mostly in language, for this is an important step in the development toward abstraction. At this level the child incorporates the other operations in a conscious way. The quotient is no longer written in the hierarchical colors, because by this time, the concept should be firm in the child's mind.

```
7687 ÷ 5 =   _____
            5|7687
```

Set up the materials as before. We must see how many times this group of 5 (indicate the skittles) is contained in this 7 (the units of thousands). Distribute the beads. Record the quotient. One group of 5, that is 5 x 1 = 5. Write 5 under the 7 and subtract. This is our remainder. Check to see if the number of beads in the bowl matches the difference.

Change the remaining two units of thousands to hundreds and bring down the hundreds.
Now we must find how many groups of 5 are contained in 26.

Age: from 6 - 7 years (at 7 years old, the child should reach abstraction)

Note: When the child has reached abstraction of division, he has actually reachedabstraction for all the operations, since division involves all of the operations. Before progressing to Big Division (having a divisor of more than one digit) the child should have reached abstraction with small division, that is, without the materials.

BIG DIVISION (TWO OR MORE DIGITS IN THE DIVISOR)

Introduction:
Recall with the child the presentation of the concept of division with a two-digit divisor, using the decimal system materials and the arm ribbons. Introduce the bead board for tens. Recall that each blue skittle represents 10 units. If I give 10 to the blue skittle, what must I give to the green skittle? After this concept is already recalled, begin division.

1st level:

```
37,464 ÷ 24 =    ______
              24|37464
```

Set up the material as before. Bring down the tens and units of the thousands, one for each board. Distribute the beads: one 10,000 for the tens; one thousand for the units. The first digit of the quotient is 1, but 1 what? The result is what one unit receives, so it is one thousand. Record the digit in color.

```
37,464 ÷ 24 =     __1___
            1  24|37464
```

Remove the beads from the board. Change the 10,000 to ten units of thousands and out away the ten thousands. Move the rack and bowl of units of thousands to the left, to the tens board. Bring down the hundreds. Distribute.
When the bowl of the lesser of the two hierarchies being considered is emptied, continue changing and distributing. However, when the bowl of the greater hierarchy is emptied, we must stop, record the digit of the quotient, and move on to another hierarchy.

```
37,464 ÷ 24 =        _1561
         1561   24|37464
```

2nd level:

7886 ÷ 35 = 35|$\overline{7886}$

As for the second level of the small division, record the remainder and bring down the digits of the dividend.

3rd level-Group Division with a two digit divisor

Recall the meaning of group division in the same way as before.

8847 ÷ 24 = 24|$\overline{8847}$

Set up the problem as usual. Bring down the first two hierarchies. How many times is 24 contained in 88? First, we must find how many times 2 is contained in 8. Distribute the beads 4 times. Now we must see if 4 is contained in 8, 4 times also. Distribute the beads. It doesn't work. Take off one row of beads from the board and place them in the bowl. Change a thousand bead to 10 hundreds and distribute.
Since we want to make groups of 24, there must be the same number of groups of 2, as there are groups of 4. Thus 3 groups of 24 were made. 3 what? Refer to what one unit received - 3 hundreds. Multiply 3 x 24, carrying mentally and recording the product beneath 88 in the dividend. (The number that we had to carry in the small multiplication, corresponds to the number of changes that were made while distributing.) Subtract; the difference should match the quantity that remained in the bowls. Remove the beads. Change. Bring down a new hierarchy and continue as before.

LONG DIVISION WITH A THREE DIGIT DIVISOR

Note: If the child has reached abstraction in division with a 2-digit divisor, he will encounter very little difficulty here because the mechanics of the operation are the same. Thus, the material will be used less by the child. The material is used for the presentation to be certain that the child has understood the concept. At this level, group division is used immediately.

Presentation:
Recall the activity done with the decimal system material and arm ribbons. Note the difference that the centurion received over the decurion and the unit. Present the materials.

56,438 ÷ 234 = 241 r.44

241 r. 44
234|56438

The procedure follows the pattern set down previously, now using 3 bead boards, and bringing down 3 hierarchies at a time. Remember that the first digit tells what all of the others must receive: How many times is 2 contained in 5? 2 We must see if 3 is contained in 6 2 times and if 4 is contained in 4 2 times.

DIVISION WITH ZERO IN THE DIVISOR

51,252 ÷ 207 =

Recall the similar case in the stamp game where a counter took the place of zero for the skittles. Here the board without any skittles reminds us of the zero. Move down the three hierarchies- one for each board. The hierarchy by the empty board reminds us of what the tens would receive if there were any. Distribute as before using group division.

19,293 ÷ 370 = 676 ÷ 300 =

Make the child conscious of what the units would have received, if there were any , in order to determine the value of the digit of the quotient. Hierarchic colors can be used for recording the quotient.

Dividend: 70,569 ÷ 229 =

Place the dividend in the bowls, leaving one empty. Do not put it back on the tray since it will be needed for making changes. Bring down the hierarchies as usual, ignoring the fact that one bowl is empty; that hierarchy corresponds to one of the digits in the divisor.

Age: 9 years (by 9 1/2 the child should reach abstraction)

Aims:
...mastery of long division
...knowledge of the reasons for every aspect of the procedure

Word Problems

FROM COMBINATION CARDS

The teacher prepares seven special combination cards and mixes them with the other 21. The child fishes for one, solves it, writing it down substituting the ? for the answer in red.
Also, word problems are prepared and mixed with the others. The child copies the text of the word problem, writes an equation with the answer in red, and writes a conclusion, that is, a complete sentence which answers the question stated in the word problem.

Aim: further understanding of the concept of multiplication

DISTRIBUTIVE VS. GROUP DIVISION

These word problems are to aid the child's understanding of the difference between distributive division and group division:

Example: (distributive) Mother has 24 cookies. She wants to give these out to her three children equally. How many cookies will she give to each child?

Example: (group) Mother has 24 cookies. She wants to make up packages with three cookies in each package. How many packages can she make?

Divisibility

DIVISIBILITY BY 2

Materials:
...decimal system materials (wooden)
...blackboard or blank chart

Presentation:
Write a number and invite the child to bring that quantity with the materials. Try to make two equal groups from this quantity, making changes as necessary .If it is possible to make two equal groups, write "yes" next to the number and underline the last digit. If not, write "no". Add or subtract one unit, and repeat the process. Examine many numbers in the same way. At a certain point the child will realize the rule: **When** the **last** digit **is** an **even** number **or** zero, **the** number **is** divisible **by** 2. If the child does not reach this point of consciousness on her own, ask questions to call her attention to the relationship between the yes or no and the oddness or evenness of the last digit.

1126 yes
1125 no
1124 yes
1123 no
1122 yes
78 yes
79 no
80 yes
12 yes

DIVISIBILITY BY 4

Materials: Same as above

Presentation:
The procedure is the same as before, except that the last two digits are underlined. Now it is no longer a matter of oddness or eveness.
Rule: When the last two digits are divisible by 4, or they are both zeros, the number is divisible by 4.

816 yes
817 no
818 no
819 no
820 yes

DIVISIBILITY BY 5

Materials: Same as above

Presentation:
The procedure is the same as before, only the last digit is underlined.
Rule: When the last digit is 5 or 0, the number is divisible by 5.

125 yes
126 no
127 no
130 yes

4<u>5</u> yes
10<u>0</u> yes

DIVISIBILITY BY 25

Materials:
...ten bars and 40 or more golden unit beads
...small white square pieces of paper,
...pen

Presentation:
Put out one group of 25 with a little card over it. Place another group next to it in an inverse position to make it easy to visualize the group as 50. Place a little card over it ("50"). Continue placing groups of 25 (7 tens, 5 units for 75) to the left of the previous group, the respective card is placed over the top. Continue up to 8 groups of 25, substituting 100 squares after 100.

Invite the child to speculate on the next few multiples of 25.

Rule: When the last two digits of a number are 23, 50, 75 or two zeros, the number is divisible by 25.

DIVISIBILITY BY 9

Materials:
...Peg board
...Box of pegs in hierarchic colors
...Small white square pegs
...pen

Presentation:
9 is a very important number, Why? It is the last digit that can remain loose in our system. It is the square of 3, and 3 is a perfect number. Therefore in this work we will consider 9 as the square of 3. On the peg board use 9 green pegs to construct a square of 3 by 3. This is a square of 3. Dissolve the square into a column at the top left corner of the board, labeling it 9.
Form two more squares of 3 side by side. Remove one unit from one square and add it to the other. Now one group has ten, change the ten green unit pegs for one blue ten peg. Dissolve the pegs into two columns next to the previous column. Label 18.
Repeat the procedure with 3 squares. Add one peg to each of the first two squares, taking the pegs from the last square. Change each group of 10 to a blue peg and dissolve. Continue in this way up to 10 squares = 90.
Observe that as the units decrease, the tens increase. At one extreme there are 9 units, at the other, 9 tens. This special pattern exists only in the table of 9. The sum of each pair of digits is 9. i.e. 27, 2 = 7 9.
Rule: A number is divisible by 9 when the sum of its digits equals 9 or a multiple of 9. If a number is divisible by 9, it is also divisible by 3.

Note: This characteristic may have been noticed in the multiplication booklet for memorization.

PROOF WITH MULTIPLICATION WITH RULE OF DIVISIBILITY BY 9

Materials: Decimal system materials

Presentation:
Take the thousand cube and try to make 9 equal groups. 1000 is not divisible by 9, but take away one unit and try again. 999 is divisible by 9. Write 1000 1 = 999. Repeat the procedure for 100 and 10. Write 100 1 = 99, 10 1 = 9. Choose a number: 5643. Convert it to expanded notation:

3			r.3
40 = 4 x 10	10 1 = 9	40 - 4 = 36	r.4
600 = 6 x 100	100 1 = 99	600 - 6 = 594	r.6
5000 = 5 x 1000	1000 1 = 999	5000 - 5 = 4995	+ r.5
			18

10 1 = 9, but we have 4 tens. We must multiply the whole equation by 4, which gives us 36. 36 from our original 40 leaves us 4 as a remainder. Continue for the others. Add all the remainders in the end. Since 18 is divisible by 9, the whole number is divisible by 9. (cont.)
Show an example of multiplication: 643 x 1527 = 981,861. Make sums of 9 in the digits - 643 x 1527 = 981,861. Total the remaining digits (4 from the multiplicand, 1 and 5 from the multiplicand) multiply the sums (6 x 4 = 24), and add the digits of the product
(2 + 4 = 6). This number should correspond to the sum of the remaining digits in the original product - 6. 6 is also the remainder when the original product (981,861 ÷ 9 = 109,095 r. 6) is divided by 9. The "remaining sums" from above multiplicand and multiplier (4 and 6) will also be the remainder when divided by 9: (643 ÷ 9 = 71 r. 4 and 1527 ÷ 9 = 169 r. 6).

Age: Ten years

Synthesis of Multiples & Divisibility

SYNTHESIS OF MULTIPLES AND DIVISIBILITY

Refer back to the multiples work and the charts constructed to find the multiples of numbers up to 10 (circling the numbers in different colors). Repeat this work making new observations-i.e. A number is a multiple of (is divisible by) 6 if it is also a multiple of 2 and 3. This is noticed when the charts for 2, 3, and 6 are done simultaneously. A multiple of 6 intersects the lines of multiples of 2 and 3.
This work really shows the close relationship of multiples and divisibility. Knowledge of one reinforces the other.

Take Table C with the prime factors. Here also we can find, for example, by what numbers 18 is divisible, by making all possible combinations of the prime factors:
18 = 2 x 3 x 3.
18 is divisible by 2, 3, 6, and 9.
18 is even, thus it is divisible by 2.
1 + 8 = 9, thus 18 is divisible by 9, which means it is also divisible by 3.
Since 18 is divisible by 2 and 3, it is also divisible by 6.